

L^2 -MODULI SPACES OF SYMPLECTIC VORTICES ON RIEMANN SURFACES WITH CYLINDRICAL ENDS

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ABSTRACT. Let (X, ω) be a compact symplectic manifold with a Hamiltonian action of a compact Lie group G and $\mu : X \rightarrow \mathfrak{g}$ be its moment map. In this paper, we study the L^2 -moduli spaces of symplectic vortices on Riemann surfaces with cylindrical ends. We studied a circle-valued action functional whose gradient flow equation corresponds to the symplectic vortex equations on a cylinder $S^1 \times \mathbb{R}$. Assume that 0 is a regular value of the moment map μ , we show that the functional is of Bott-Morse type and its critical points of the functional form twisted sectors of the symplectic reduction (the symplectic orbifold $[\mu^{-1}(0)/G]$). We show that any gradient flow lines approaches its limit point exponentially fast. Fredholm theory and compactness property are then established for the L^2 -Moduli spaces of symplectic vortices on Riemann surfaces with cylindrical ends.

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1. INTRODUCTION AND STATEMENTS OF MAIN THEOREMS

The symplectic vortex equations on a Riemann surface Σ associated a principal G -bundle P and a Hamiltonian G -space (X, ω) , originally discovered by K. Cieliebak, A. R. Gaio, and D. A. Salamon [11], and independently by I. Mundet i Riera [31], is a system of first order partial differential equations

$$(1.1) \quad \begin{cases} \bar{\partial}_{J,A}(u) = 0 \\ *_{\Sigma} F_A + \mu(u) = 0 \end{cases}$$

for a connection A on P and a G -equivariant map $u : P \rightarrow X$. See Section 2 for an explanation of the notations involved. They are natural generalisations of the J -holomorphic equation in a symplectic manifold for G is trivial, and of the well-known Ginzburg-Landau vortices in a mathematical model of superconductors for $\Sigma = \mathbb{C}$ and $X = \mathbb{C}^n$ as a Hamiltonian $U(1)$ -space. Ginzburg-Landau vortices have been studied both from mathematicians and physicists' viewpoints. They are two-dimensional solitons, as time-independent solutions with finite energy to certain classical field equations in Abelian Higgs model, see [25] for a complete account of Ginzburg-Landau vortices.

Since the inception of these symplectic vortices, there have been steady developments in the study of the moduli spaces of symplectic vortices and their associated invariants, the so-called Hamiltonian Gromov-Witten invariants. Many fascinating conjectures have been proposed, for example see [11], [22] and [40].

As in Gromov-Witten theory, there are several main technical issues in the definition of invariants from symplectic vortices: compactification, gluing theory and regularization for the moduli spaces of symplectic vortices. There have been many works focused on the compactification issue([12],[31],[33],[40],[36]). On the one hand, when Σ is closed, X is symplectically aspherical and satisfies some convexity condition, A. R. Gaio, I. Mundet i Riera and D. A. Salamon in [12] proved compactness of the moduli space of symplectic vortices with compact support and bounded energy. On the other hand, when $G = U(1)$ and X is closed, with strong monotone conditions, I. Mundet i Riera in [31] compactified the moduli space of bounded energy symplectic vortices over a fixed closed Riemann surface. When $G = U(1)$ and X is a general compact symplectic manifold, I. Mundet i Riera and G. Tian compactified the moduli space of symplectic vortices with bounded energy over smooth curves degenerating to nodal curves. In particular, the bubbling off phenomena near nodal points are new. Energy may be lost and there are gradient flows of the moment map instead. This is not present in the usual Gromov-Witten theory, and was elegantly and carefully presented in [33]. Also, there are some studies on special models such as on the affine vortices ([40]). Based on their compactification, Mundet i Riera and Tian have a long project on defining Hamiltonian GW invariants and almost finished([34]). On the other hand, Woodward, following Mundet i Riera's approach([32]), gave an algebraic geometry approach to define gauged Gromov-Witten invariant([35]), and show its relation to Gromov-Witten invariants of $X // G$ via quantum Kirwan morphisms([35]).

In this paper, we study the moduli spaces of symplectic vortices on a Riemann surface with cylindrical ends. In particular, for a genus g Riemann surface with n -marked points, we will study the L^2 -moduli space of symplectic vortices on a Riemann surface Σ with a cylindrical end metric near each marked points. Here the energy of (A, u) , defined to be the Yang-Mills-Higgs energy functional

$$(1.2) \quad E(A, u) = \int_{\Sigma} \frac{1}{2} (|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2) \nu_{\Sigma},$$

is finite. It turns out that the Hamiltonian GW type invariants are very sensitive to the volume forms used near punctured points. For example, readers may refer to §6 for further discussion.

In Section 2, we briefly review the moduli spaces of symplectic vortices on a closed Riemann surface as developed in [11], [12] and [33]. In Section 3, we investigate the asymptotic behaviour of symplectic vortices on a half cylinder $S^1 \times \mathbb{R}^{\geq 0}$ with finite energy. For this we adapt the action functional in [18] and [19] to get a circle-valued functional whose L^2 -gradient flow equation realizes the symplectic vortex equations (1.1) on $S^1 \times \mathbb{R}^{\geq 0}$ in temporal gauge. The critical point set of this functional, modulo gauge transformations can be identified with

$$\left(\bigsqcup_{g \in G} (\mu^{-1}(0))^g \right) / G,$$

as a topological space, where the action of G is given by $h \cdot (x, g) = (h \cdot x, hgh^{-1})$. Under the assumption that 0 is a regular value of the moment map μ , so the symplectic reduced space is a symplectic orbifold

$$\mathcal{X}_0 = [\mu^{-1}(0)/G].$$

Here we use the square bracket to denote the orbifold structure arising from the locally free action of G on $\mu^{-1}(0)$. Then the critical point set is diffeomorphic to the inertia orbifold of the symplectic orbifold \mathcal{X}_0 , denoted by

$$I\mathcal{X}_0 = \bigsqcup_{(g)} \mathcal{X}_0^{(g)}$$

where (g) runs over the conjugacy class in G with non-empty fixed points in $\mu^{-1}(0)$. Note that for a non-trivial conjugacy class (g) , $\mathcal{X}_0^{(g)}$ is often called a twisted sector of \mathcal{X}_0 , which is diffeomorphic to the orbifold arising from the action of $C(g)$ on $\mu^{-1}(0)^{(g)}$. Here $C(g)$ denotes the centralizer of g in G for a representative g in the conjugacy class (g) .

Throughout this paper, we assume that 0 is a regular value of the moment map μ . Then we show that this circle-valued functional is actually of Bott-Morse type. We also establish an crucial inequality (Proposition 3.13) near each critical point. This inequality enables us to establish an exponential decay result for a symplectic vortex on $S^1 \times \mathbb{R}^{\geq 0}$ with finite energy, Cf. Theorem 3.16.

In Section 4, we study the L^2 -moduli space $\mathcal{N}_{\Sigma}(X, P)$ of symplectic vortices on a Riemann surface Σ with k -cylindrical ends, associated to a principal G -bundle and a closed Hamiltonian

G -manifold (X, ω) . Applying the asymptotic analysis in Section 3 to the cylindrical end, we get a continuous asymptotic limit map (Proposition 4.1 and Subsection 4.2)

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow (\text{Crit})^k \cong (I\mathcal{X}_0)^k.$$

The Yang-Mills-Higgs energy functional takes discrete values on $\mathcal{N}_\Sigma(X, P)$ depending on homology classes in $H_2^G(X, \mathbb{Z})$.

Fix a homology class $B \in H_2^G(X, \mathbb{Z})$, denote by $\mathcal{N}_\Sigma(X, P, B)$ the L^2 -moduli space of symplectic vortices on a Riemann surface Σ with the topological type defined by B . Then we develop the Fredholm theory for $\mathcal{N}_\Sigma(X, P, B)$ and calculate the expected dimension of the L^2 -moduli space of symplectic vortices with prescribed asymptotic behaviours.

Theorem A (Theorem 4.3) *Let $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})$ be the subset of $\mathcal{N}_\Sigma(X, P, B)$ consisting of symplectic vortices $[(A, u)]$ such that*

$$\partial_\infty(A, u) \in (\mathcal{X}_0^{(g_1)} \times \cdots \times \mathcal{X}_0^{(g_k)}) \subset (\text{Crit})^k$$

Then $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})$ admits an orbifold Fredholm system with its virtual dimension given by

$$2\langle c_1^G(TX), B \rangle + 2(n - \dim G)(1 - g_\Sigma) - 2 \sum_{i=1}^k \iota_{CR}(\mathcal{X}_0^{(g_i)}, \mathcal{X}_0)$$

where g_Σ is the genus of the Riemann surface Σ . Here $\iota_{CR}(\mathcal{X}_0^{(g_i)}, \mathcal{X}_0)$ is the degree shift as introduced in [10].

In Section 5, we also establish the compactness property for these L^2 -moduli spaces of symplectic vortices on Σ with prescribed asymptotic behaviours. We show that there are two types of limiting vortices appearing in the compactification. The first type occurs as the bubbling phenomenon of pseudo-holomorphic spheres at interior points just as in the Gromov-Witten theory. To describe this type of limiting vortices, we introduce the usual weighted trees to classify the resulting topological type. The second type is due to the sliding-off of the Yang-Mills-Higgs energy along the cylindrical ends as happened in the Floer theory. The combination of these types of convergence sequences is called the weak chain convergence in instanton Floer theory in [13]. The choice of cylindrical metric on Σ is crucial in our study the compactness property in the sense that these are only two types of limiting vortices appearing in the compactification of the L^2 -moduli spaces of symplectic vortices on a cylindrical Riemann surface.

To describe the topological types appearing in the compactification, we introduce a notion of web of stable weighted trees of the type $(\Sigma; B)$ consists of a principal tree Γ_0 with k -tails and a collections of ordered sequence of trees of finite length

$$\Gamma_i = \bigsqcup_{j=1}^m T_i(j)$$

for each tail $i = 1, \dots, k$. See Definition 5.1 for a precise definition. Let $\mathcal{S}_{\Sigma; B}$ be the set of webs of stable weighted trees of the type $(\Sigma; B)$, which is a partially ordered finite set. For each $\Gamma \in \mathcal{S}_{\Sigma; B}$, we associate an L^2 -moduli space \mathcal{N}_Γ of symplectic vortices of type Γ . Let

$\mathcal{N}_\Gamma((g_1), \dots, (g_k))$ be the corresponding L^2 -moduli space of symplectic vortices of type Γ with prescribed asymptotic data in

$$\mathcal{X}_0^{(g_1)} \times \dots \times \mathcal{X}_0^{(g_k)} \subset (\text{Crit})^k.$$

Then the main theorem of this paper is to show that the coarse L^2 -moduli space of symplectic vortices on Σ can be compactified into a stratified topological space whose strata are labelled by a web of stable weighted trees in $\mathcal{S}_{\Sigma;B}$. In the following theorem, we use the notation $|\mathcal{N}|$ to denote the coarse space of an orbifold topological space \mathcal{N} .

Theorem B (Theorem 5.5) *Let Σ be a Riemann surface of genus g with k -cylindrical ends. The coarse space L^2 -moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ can be compactified to a stratified topological space*

$$|\overline{\mathcal{N}}_\Sigma(X, P, B)| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma|$$

such that the top stratum is $|\mathcal{N}_\Sigma(X, P, B)|$. Moreover, the coarse moduli space

$$|\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})|$$

with a specified asymptotic datum can be compactified to a stratified topological space

$$|\overline{\mathcal{N}}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma((g_1), \dots, (g_k))|.$$

Remark 1.1. Note that the evaluation map has its image in $I\mathcal{X}_0$, hence our invariants will define on $H_{CR}^*(\mathcal{X}_0)$. This is different from the Hamiltonian Gromov-Witten invariants defined earlier, as the invariants are defined on $H_G^*(X)$ in [12] and [35]. Hence the invariants we will define is essentially different from the usual HGW invariants. One may refer to §6 for further discussion.

We remark that the compactness properties of the moduli spaces of symplectic vortices have been studied earlier in [33], [36], [40], [41] and [39]. Under the assumption that X is a Kähler Hamiltonian G -manifold with semi-free action, the above compactness theorem has also been obtained by Venugopalan in [39] using a different approach.

2. REVIEW OF SYMPLECTIC VORTICES

In this section, we review some of basic facts for the symplectic vortices following [12] [31] and [33].

2.1. Symplectic vortex equations.

Let (X, ω) be a $2n$ -dimensional symplectic manifold with a compatible almost complex structure J and a Hamiltonian action of a connected compact Lie group G

$$G \times X \longrightarrow X, \quad (g, x) \mapsto gx.$$

Let \mathfrak{g} be the Lie algebra of G with a G -invariant inner product $\langle \cdot, \cdot \rangle$. Recall that an action of G on M is Hamiltonian if there exists an equivariant map, called the moment map,

$$\mu : X \longrightarrow \mathfrak{g}$$

satisfying the defining property

$$d\mu_\xi = \tilde{\xi} \lrcorner \omega = \omega(\tilde{\xi}, \cdot), \quad \text{for any } \xi \in \mathfrak{g}.$$

Here the function μ_ξ is given by $\mu_\xi(x) = \langle \mu(x), \xi \rangle$, and $\tilde{\xi}$ is the vector field on X defined by the infinitesimal action of $\xi \in \mathfrak{g}$ on X

$$(\tilde{\xi}f)(x) = \frac{d}{dt}f(\exp(-t\xi)x), \quad \text{for } f \in C^\infty(X),$$

and the symbol \lrcorner denotes contraction between differential forms and vector fields. Note that the moment map is unique up to a shift by an element $\tau \in Z(\mathfrak{g})$ (the centre Lie subalgebra of \mathfrak{g}). See Chapter 2 in [24] for detailed a discussion on the geometry of moment maps.

Let $P \rightarrow \Sigma$ be smooth (principal) G -bundle over a Riemann surface (Σ, j_Σ) (not necessarily compact and possibly with boundary). Let g_Σ be a Riemannian metric on Σ and $(*_\Sigma, \nu_\Sigma)$ be the associated Hodge star operator and volume form. Denote by $C_G^\infty(P, X)$ be the space of smooth G -equivariant maps $u : P \rightarrow X$ and by $\mathcal{A}(P)$ the space of connections on P which is an affine space modelled $\Omega^1(\Sigma, P^{ad})$. Here $P^{ad} = P \times_G \mathfrak{g}$ is the bundle of Lie algebras associated to the adjoint representation $ad : G \rightarrow GL(\mathfrak{g})$.

Denote the associated fiber bundle of P by

$$\pi : Y = P \times_G X \longrightarrow \Sigma$$

the symplectic fiber bundle. Then a smooth G -equivariant map $u : P \rightarrow X$ yields a section $\tilde{u} : \Sigma \rightarrow Y$. Note that any connection A on P induces splittings

$$TP \cong \pi^*T\Sigma \oplus T^{\text{vert}}P, \quad TY \cong \pi^*T\Sigma \oplus T^{\text{vert}}Y.$$

The covariant derivative $d_A \tilde{u} \in \Omega^1(\Sigma, \tilde{u}^*T^{\text{vert}}Y)$ is derived from du as follows:

$$d_A u : \pi^*T\Sigma \xrightarrow{du} TY \xrightarrow{\text{projection}} T^{\text{vert}}Y.$$

For simplicity, we denote $d_A \tilde{u}$ by $d_A u$ as well.

The **symplectic vortex equations** on Σ are the following first order partial differential equations for pairs $(A, u) \in \mathcal{A}(P) \times C_G^\infty(P, X)$

$$(2.1) \quad \begin{cases} \bar{\partial}_{J,A}(u) = 0 \\ *_\Sigma F_A + \mu(u) = 0 \end{cases}$$

where F_A is the curvature of the connection A . The almost complex structures j_Σ and J define an almost complex structure J_A on Y . The first equation in (2.1) implies that \tilde{u} is a J_A -holomorphic section. In term of the covariant derivative $d_A u \in \Omega^1(\Sigma, u^*T^{\text{vert}}Y)$, it is given by

$$(2.2) \quad \bar{\partial}_{J,A}(u) = \frac{1}{2}(d_A u + J \circ d_A u \circ j_\Sigma) = 0$$

in $\Omega^{0,1}(\Sigma, u^*T^{\text{vert}}Y)$. For the second equation in (2.1), we remark that $\mu \circ u$ is a section of P^{ad} and the Hodge star operator defines a map

$$*_\Sigma : \Omega^2(\Sigma, P^{ad}) \longrightarrow \Omega^0(\Sigma, P^{ad}).$$

Using the Riemannian volume ν_Σ , the second equation in (2.1) can be written as

$$(2.3) \quad F_A + \mu(u)\nu_\Sigma = 0.$$

A solution (A, u) to (2.1) is called a symplectic vortex on Σ associated to a principal G -bundle P and a Hamiltonian G -space X . Two elements $w = (P, A, u)$ and $w' = (P', A', u')$ are called equivalent iff there is a bundle isomorphism

$$\Phi : P' \rightarrow P$$

such that

$$\Phi^*(A, u) = (\Phi^*A, u \circ \Phi) = (A', u').$$

When P is evident in the context, we will omit P from the notation and simply call (A, u) for a symplectic vortex on Σ . As the symplectic vortex equations (2.1) on Σ for a fixed P is invariant under the action of gauge group $\mathcal{G}(P) = \text{Aut}(P)$, the moduli space of symplectic vortices on Σ is the set of solutions to (2.1) modulo the gauge transformations. We remark that P is an essential part of symplectic vortices, in particularly in the study of the compactifications of the moduli spaces of vortices.

Given $u : P \rightarrow X$, there is an equivariant classifying map $P \rightarrow EG$. Together with $u : P \rightarrow X$, they define to a continuous map

$$u_G : \Sigma \rightarrow X_G := EG \times_G X,$$

which in turn determines a degree 2 equivariant homology class $[u_G]$ in $H_2^G(X, \mathbb{Z})$ (if Σ is closed). Denote by $\widetilde{\mathcal{M}}_\Sigma(X, B)$ the space of symplectic vortices on Σ associated (P, X) with a fixed equivariant homology class in $B \in H_2^G(X, \mathbb{Z})$, that means,

$$\widetilde{\mathcal{M}}_\Sigma(X, B) = \{(A, u) | [u_G] = B, (A, u) \text{ satisfies the equations (2.1)}\}.$$

The quotient of $\widetilde{\mathcal{M}}_\Sigma(X, B)$ under the gauge group $\mathcal{G}(P)$ -action

$$\mathcal{M}_\Sigma(X, B) = \widetilde{\mathcal{M}}_\Sigma(X, B) / \mathcal{G}(P)$$

is called the moduli space of symplectic vortices with a fixed homology class B .

A solution to (2.1) with a fixed $B \in H_2^G(X, \mathbb{Z})$ is an absolute minimizer (hence, a critical point) of the Yang-Mills-Higgs energy functional

$$(2.4) \quad E(A, u) = \int_\Sigma \frac{1}{2} (|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2) \nu_\Sigma.$$

This is due to the fact (Proposition 3.1 in [11]) that for any $(A, u) \in \mathcal{A}(P) \times C_G^\infty(P, X)$,

$$(2.5) \quad E(A, u) = \int_\Sigma \left(|\bar{\partial}_{J,A}(u)|^2 + \frac{1}{2} |*_\Sigma F_A + \mu(u)|^2 \right) \nu_\Sigma + \int_\Sigma u^* \omega - d\langle \mu(u), A \rangle.$$

Here, $u^* \omega - d\langle \mu(u), A \rangle$ is a horizontal and G -equivariant 2-form on P and descends to a 2-form Σ , denoted by the same notation. On the other hand, $[\omega - \mu] \in H_2^G(X)$ is the equivariant

cohomology class defined by the equivariant closed 2-form $\omega - \mu \in \Omega_G^2(X)$. The pairing $\langle [\omega - \mu], [u_G] \rangle$ is computed by

$$\langle [\omega - \mu], [u_G] \rangle = \int_{\Sigma} ((d_A u)^* \omega - \langle \mu(u), F_A \rangle) = \int_{\Sigma} u^* \omega - d \langle \mu(u), A \rangle.$$

Here $d_A u$ is a horizontal and G -equivariant one-form on P with values in u^*TX and descends from a $u^*T^{\text{vert}}Y$ -valued one form on Σ , see Proposition 3.1 in [11]. The Yang-Mills-Higgs energy functional (2.4) and the identity play vital roles in the study of the moduli space $\mathcal{M}_{\Sigma}(X, B)$.

Remark 2.1. We remark that (2.5) is true for any surface Σ . In particular, when (A, u) is a symplectic vortex on Σ ,

$$(2.6) \quad E(A, u) = \int_{\Sigma} u^* \omega - d \langle \mu(u), A \rangle.$$

This is the crucial identity for us to define the action functional \mathcal{L} in Section 3.

Remark 2.2. (1) If $G = U(1)$ the unit circle in \mathbb{C} and $X = \mathbb{C}^n$ with the usual action of $U(1)$ by multiplication, then symplectic vortex equation is a generalisation of the well-studied vortex equations (Cf. [25]). In particular, when Σ is compact and $X = \mathbb{C}$, Bradlow and Garcia-Prada showed that the moduli space of vortices on Σ with vortex number

$$N = \langle c_1(P \times_{U(1)} \mathbb{C}), [\Sigma] \rangle$$

is empty if $N > \text{Vol}(\Sigma)/4\pi$, and is $\text{Sym}^N(\Sigma)$, the N -th symmetric product of Σ , if $N > \text{Vol}(\Sigma)/4\pi$.

(2) As observed in [11], the space $\mathcal{A}(P) \times C_G^{\infty}(P, X)$ is an infinite dimensional Fréchet manifold with a natural symplectic structure. The action of gauge group $\mathcal{G}(P)$ is Hamiltonian with a moment map

$$\mathcal{A}(P) \times C_G^{\infty}(P, X) \rightarrow C_G^{\infty}(\Sigma, P^{ad})$$

defined by $(A, u) \mapsto *F_A + \mu(u)$. Hence, the moduli space of symplectic vortices can be thought as a symplectic quotient if the space

$$\mathcal{S} = \{(A, u) | \bar{\partial}_{J,A}(u) = 0\}$$

is a symplectic submanifold of $\mathcal{A}(P) \times C_G^{\infty}(P, X)$. In practice, the space \mathcal{S} is not a smooth submanifold in general. It still provides a good guiding principle for the development of Hamiltonian Gromov-Witten theory. See [1] [3] for some applications of this principle in similar contexts.

When $\Sigma = S^1 \times \mathbb{R}$ with the flat metric $(dt)^2 + (d\theta)^2$ and the standard complex structure $j(\partial_t) = \partial_{\theta}$, with respect to a fixed trivialisation of P , we can use the temporal gauge

$$A = d + \xi(\theta, t)d\theta, \quad \text{for } \xi : S^1 \times \mathbb{R} \rightarrow \mathfrak{g},$$

to write the symplectic vortex equations (2.1) as

$$(2.7) \quad \begin{cases} \frac{\partial u}{\partial t} + J \left(\frac{\partial u}{\partial \theta} + \tilde{\xi}(\theta, t)(u(x)) \right) = 0 \\ \frac{\partial \xi}{\partial t} + \mu(u) = 0. \end{cases}$$

This is the downward gradient flow equation for a particular function on $C^\infty(S^1, X \times \mathfrak{g})$ defined in Section 3, where we will study this function in more details.

2.2. Moduli spaces of symplectic vortices on a Riemann surface.

In the study of the moduli space $\mathcal{M}_\Sigma(X, B)$, we need to develop certain Fredholm theory. This requires some Sobolev completion of the space

$$\tilde{\mathcal{B}} = \mathcal{A}(P) \times C_{G,B}^\infty(P, X)$$

where $C_{G,B}^\infty(P, X) = \{u \in C_G^\infty(P, X) \mid [u_G] = B\}$. The Sobolev embedding theorem in dimension 2 leads to the $W^{1,p}$ Sobolev spaces for $p > 2$. Depending the question at hand regarding Riemann surface Σ being closed, cylindrical or asymptotically Euclidean at infinite, further careful choice of a suitable Sobolev spaces is needed. Instead, in this subsection, we only review the linearization of the symplectic vortex equations and the gauge transformations on $\tilde{\mathcal{B}}$ with the Fréchet topology as in [11].

Let $\tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{B}}$ be the $\mathcal{G}(P)$ -equivariant vector bundle whose fiber over (A, u) is given by

$$\tilde{\mathcal{E}}_{(A,u)} = \Omega^{0,1}(\Sigma, u^*T^{\text{vert}}Y) \oplus \Omega^0(\Sigma, P^{ad}).$$

Then the symplectic vortex equations (2.1) defines a $\mathcal{G}(P)$ -equivariant section

$$S(A, u) = (\bar{\partial}_{J,A}(u), *_\Sigma F_A + \mu(u))$$

such that $\tilde{\mathcal{M}}_\Sigma(X, B)$ is the zeros of this section. The vertical differential of this section, denoted by $\mathcal{D}_{A,u}$, together with the linearization $L_{A,u}$ of the gauge transformation at $(A, u) \in S^{-1}(0)$ give rise to the deformation complex

$$\Omega^0(\Sigma, P^{ad}) \xrightarrow{L_{A,u}} \Omega^1(\Sigma, P^{ad}) \oplus \Omega^0(\Sigma, u^*T^{\text{vert}}Y) \xrightarrow{\mathcal{D}_{A,u}} \Omega^{0,1}(\Sigma, u^*T^{\text{vert}}Y) \oplus \Omega^0(\Sigma, P^{ad}).$$

Here $L_{A,u}$ is given by $L_{A,u}(\eta) = (-d_A\eta, \tilde{\eta}(u))$, and $\mathcal{D}_{A,u}$ is the linearization operator of the symplectic vortex equations (2.1).

If Σ is closed, using the usual $W^{1,p}$ -Sobolev space for $p > 2$, it was shown in [12] that the operator $\mathcal{D}_{A,u} \oplus L_{A,u}^*$ is a Fredholm operator for any $W^{1,p}$ -pair (A, u) with real index given by

$$(n - \dim G)\chi(\Sigma) + 2\langle u^*(c_1(T^{\text{vert}}Y)), [\Sigma] \rangle.$$

Equivalently, by completing $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{B}}$ under the $W^{1,p}$ and L^p -norms respectively, the triple $(\tilde{\mathcal{B}}, \tilde{\mathcal{E}}, S)$ defines a Fredholm system

$$(\mathcal{B}, \mathcal{E}, S)$$

after modulo the $W^{2,p}$ gauge transformation group. The zero set of S is the moduli space $\mathcal{M}_\Sigma(X, B)$. The central issue in the study of the moduli space of symplectic vortices is to

establish a virtual fundamental cycles as in [20] or a virtual system as in [7] for a compactified moduli space $\mathcal{M}_\Sigma(X, B)$.

3. SYMPLECTIC VORTICES ON A CYLINDER $S^1 \times \mathbb{R}$

The symplectic vortex equations (2.7) on $S^1 \times \mathbb{R}$ in temporal gauge suggests that it is a gradient flow equation for an action functional on an infinite dimensional space \mathcal{C} . This functional has been studied in [11], [18] and [41]. After we studied the critical point set and the Hessian of this functional, we establish an inequality (Proposition 3.13) for this functional which plays a crucial role in analysing the asymptotic behaviour of an L^2 symplectic vortex on $S^1 \times [0, \infty)$. This crucial inequality is applied to show that a gradient flow line γ with a finite energy condition

$$E(\gamma) = \int_0^\infty \left\| \frac{\partial \gamma(t)}{\partial t} \right\|^2 dt < \infty$$

has a well-defined limit point, and converges exponentially fast to the limit point. Similar exponential decay estimates has also been obtained by in [33] and [41] using different methods.

3.1. Action functional for symplectic vortices. Let P_{S^1} be a principal G -bundle over S^1 , and \mathcal{A}_{S^1} be the space of smooth connections on P_{S^1} which is an affine space over $\Omega^1(S^1, \mathfrak{g})$. Since $C_G^\infty(P_{S^1}, X) \cong C^\infty(S^1, X_G)$, the connected component of \mathcal{C} is identified with $\pi_1^G(X)$. For each $c \in \pi_1^G(X)$ we denote the component by \mathcal{C}^c .

Now choosing a trivialisation $P_{S^1} \rightarrow S^1 \times G$ and the standard metric from $S^1 \cong \mathbb{R}/\mathbb{Z}$, we have the identification

$$\mathcal{C} = C_G^\infty(P_{S^1}, X) \times \mathcal{A}_{S^1} \cong C^\infty(S^1, X \times \mathfrak{g}).$$

We sometimes use the same notation to denote both a map in $C_G^\infty(P_{S^1}, X)$ and in $C^\infty(S^1, X)$ which should be clear in the context. We remark that the identification of \mathcal{A}_{S^1} with $C^\infty(S^1, \mathfrak{g})$ is with respect to the trivial connection on P_{S^1} .

With respect to the Fréchet topology, \mathcal{C} is a smooth manifold whose tangent space at (x, η) is

$$T_{(x, \eta)}\mathcal{C} = \Gamma_{C^\infty}(S^1, x^*TX \times \mathfrak{g}),$$

the space of smooth sections of the bundle $x^*TX \times \mathfrak{g}$. Under the identification $\mathcal{C} = C^\infty(S^1, X \times \mathfrak{g})$, the full gauge group $LG = C^\infty(S^1, G)$ acts on \mathcal{C} by

$$(3.1) \quad g \cdot (x, \eta) = (gx, g^{-1} \frac{dg}{d\theta} + g^{-1} \eta g).$$

Here we simply denote by $g^{-1} \eta g$ the adjoint action g^{-1} on η .

Let (x_0, η_0) and (x_1, η_1) be in a connected component of \mathcal{C} and γ be a path

$$\gamma(t) = (x(t), \eta(t)) : I = [0, 1] \rightarrow \mathcal{C}$$

connecting (x_0, η_0) and (x_1, η_1) . Then γ determines a pair

$$(u_\gamma, A_\gamma) \in C_G^\infty(P_{S^1} \times I, X) \times \mathcal{A}(P_{S^1} \times I).$$

Define the energy functional for this path γ as

$$(3.2) \quad E(\gamma) = \int_{S^1 \times I} ((d_{A_\gamma} u_\gamma)^* \omega - \langle \mu(u_\gamma), F_{A_\gamma} \rangle).$$

Note that if the path γ satisfies the symplectic vortex equations (2.7) on $[0, 1] \times S^1$, then $E(\gamma)$ agrees with its Yang-Mills-Higgs energy. Using the coordinate (θ, t) for $S^1 \times I$, we can compute (cf. (2.6))

$$E(\gamma) = - \int_{S^1 \times I} (x(t))^* \omega + \int_{S^1} (\langle \mu(x_0), \eta_0 \rangle - \langle \mu(x_1), \eta_1 \rangle) d\theta.$$

Lemma 3.1. *Under the identification $C_G^\infty(P_{S^1} \times I, X) \times \mathcal{A}(P_{S^1} \times I) \cong C^\infty(S^1 \times I, X \times \mathfrak{g})$, the energy function defined in (3.2) enjoys the following properties.*

(1) *For any $g \in LG$, let $g \cdot \gamma$ be the path obtained from the action of g , then*

$$E(\gamma) = E(g \cdot \gamma).$$

(2) *If γ_1 and γ_2 are homotopic paths relative to the boundary point (x_0, η_0) and (x_1, η_1) , then $E(\gamma_1) = E(\gamma_2)$.*

Proof. (1) is obvious. We explain (2). The path $\gamma_1 \# (-\gamma_2)$ defines a pair (u, A) on a bundle P over $S^1 \times S^1$, then

$$E(\gamma_1) - E(\gamma_2) = \langle [\omega - \mu], [u_G] \rangle.$$

Since $\gamma_1 \sim \gamma_2$, $[u_G] = 0$. Hence $E(\gamma_1) = E(\gamma_2)$. \square

We now define a (circle-valued) function on \mathcal{C} as follows. For each component \mathcal{C}^c we fix a based point (x_c, η_c) . Given a point $(x, \eta) \in \mathcal{C}^c$, let $\gamma : [0, 1] \rightarrow \mathcal{C}^c$ be a path connecting (x_c, η_c) and (x, η) . As above, this can be written as a pair

$$(\tilde{x}, \tilde{\eta}) \in C_G^\infty(P_\Sigma, X) \times \mathcal{A}_\Sigma,$$

where $\Sigma = [0, 1] \times S^1$ and \mathcal{A}_Σ is the space of connections on a principal G -bundle $P_\Sigma = P_{S^1} \times [0, 1]$. Then we define

$$(3.3) \quad \mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta}) = E(\tilde{x}, \tilde{\eta}).$$

For a different extension $(\Sigma', \tilde{x}', \tilde{\eta}')$, by the same argument in the proof of Lemma 3.1, we know that

$$\mathcal{L}_{\Sigma'}(\tilde{x}', \tilde{\eta}') - \mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta}) = \langle [\omega - \mu], [u_G] \rangle,$$

for some $[u_G] \in H_2^G(X, \mathbb{Z})$ defined by x and x' . If $(\tilde{x} \# \tilde{x}', \tilde{\eta} \# \tilde{\eta}')$ is not smooth, we can choose a smooth pair which is homotopic to $(\tilde{x} \# \tilde{x}', \tilde{\eta} \# \tilde{\eta}')$. Then the topological invariance ensures that the result does not depend on the choice of the smooth pair. Recall that $\langle [\omega - \mu], \cdot \rangle$ is the homomorphism

$$\langle [\omega - \mu], \cdot \rangle : H_2^G(X, \mathbb{Z}) \longrightarrow \mathbb{R}.$$

The image of $\langle [\omega - \mu], \cdot \rangle$ consists of integer multiples of a fixed positive real number $N_{[\omega - \mu]}$. Hence, modulo $\mathbb{Z}N_{[\omega - \mu]}$, $\mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta})$ descends to a well-defined function

$$(3.4) \quad \mathcal{L}(x, \eta) = \mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta}) \pmod{\mathbb{Z}N_{[\omega - \mu]}}.$$

We denote by $\mathcal{L} : \mathcal{C} \rightarrow \mathbb{R}/\mathbb{Z}N_{[\omega-\mu]}$ the resulting circle-valued function.

Lemma 3.1 implies that the following action functional on \mathcal{C} is well-defined.

Definition 3.2. Given a collection of base points $\{(x_c, \eta_c) | c \in \pi_1^G(X)\}$ for the connected components \mathcal{C} labelled by $\pi_1^G(X)$, let $\tilde{\mathcal{C}}_{uni}$ be the associated universal cover of \mathcal{C} defined by the homotopy paths. The action functional on $\tilde{\mathcal{C}} : \tilde{\mathcal{C}}_{uni} \rightarrow \mathbb{R}$ is defined by (3.3) for a homotopy path from $(x, \eta) \in \mathcal{C}$ to the base point for the connected component. The induced function

$$\mathcal{L} : \mathcal{C} \longrightarrow \mathbb{R}/\mathbb{Z}N_{[\omega-\mu]}$$

is called the action functional on \mathcal{C} .

Remark 3.3. There is a minimal covering space of \mathcal{C} , denoted by $\tilde{\mathcal{C}}$, such that the action functional \mathcal{L} can be lifted to a \mathbb{R} -valued function $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}$ and the following diagram commutes

$$(3.5) \quad \begin{array}{ccc} \tilde{\mathcal{C}}_{uni} & \xrightarrow{\tilde{\mathcal{L}}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{C}} & \xrightarrow{\tilde{\mathcal{L}}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\mathcal{L}} & \mathbb{R}/\mathbb{Z}N_{[\omega-\mu]}. \end{array}$$

We write an element of $\tilde{\mathcal{C}}$ in the fiber over $(x, \eta) \in \mathcal{C}$ as an equivalent class a path connecting (x, η) to the base point of the connected component.

As the covering map $\tilde{\mathcal{C}}_{uni} \rightarrow \mathcal{C}$ is a local diffeomorphism, the differential and the Hessian operator of \mathcal{L} can be calculated by the Fréchet derivatives of $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}_{uni}$ or $\tilde{\mathcal{C}}$. For this purpose, we introduce an L^2 -inner product on the tangent bundle $T\mathcal{C}$, that is, for $(v_1, \xi_1), (v_2, \xi_2) \in T_{(x, \eta)}\mathcal{C}$,

$$(3.6) \quad \langle (v_1, \xi_1), (v_2, \xi_2) \rangle = \int_{S^1} (\omega(v_1, Jv_2) + \langle \xi_1, \xi_2 \rangle) d\theta.$$

Proposition 3.4. *With respect to the L^2 -inner product, the L^2 -gradient of \mathcal{L} is given by*

$$(3.7) \quad \nabla \mathcal{L}(x, \eta) = \left(J\left(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x\right), \mu(x) \right).$$

Hence, the critical point set is define by the equations

$$(3.8) \quad \frac{\partial x}{\partial \theta} + \tilde{\eta}_x = 0, \quad \mu(x) = 0.$$

Proof. Let $(D\mathcal{L})_{(x,\eta)}$ be the first Fréchet derivative of \mathcal{L} , that is, for any $(v, \xi) \in T_{(x,\eta)}\mathcal{C}$,

$$\begin{aligned}
 (D\mathcal{L})_{(x,\eta)}(v, \xi) &= \left. \frac{\partial}{\partial s} \right|_{s=0} \mathcal{L}(\exp_x(sv), \eta + s\xi) \\
 &= - \int_{S^1} \omega(v, \frac{\partial x}{\partial \theta}) d\theta + \int_{S^1} (\langle d\mu_x(v), \eta \rangle + \langle \mu(x), \xi \rangle) d\theta \\
 (3.9) \quad &= \int_{S^1} \left(\omega(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x, v) + \langle \mu(x), \xi \rangle \right) d\theta \\
 &= \int_{S^1} \left(\omega(J(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x), Jv) + \langle \mu(x), \xi \rangle \right) d\theta \\
 &= \langle (J(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x), \mu(x)), (v, \xi) \rangle.
 \end{aligned}$$

Hence, L^2 -gradient of \mathcal{L} at (x, η) is given by (3.7). The proposition is proved. \square

Remark 3.5. The gradient equation $\nabla \mathcal{L}(x, \eta) = 0$ can be thought as the Euler-Lagrange equations for the action functional \mathcal{L} . Moreover, the downward gradient flow equation of \mathcal{L} on \mathcal{C}

$$(3.10) \quad \frac{\partial}{\partial t} (x(t), \eta(t)) = - \left(J(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x), \mu(x) \right)$$

is exactly the symplectic vortex equation (2.7) on $S^1 \times \mathbb{R}$ in temporal gauge.

Before we proceed further, let us investigate the gauge invariance of the action functional $\tilde{\mathcal{L}}$.

Lemma 3.6. *The action functional $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}$ is invariant under the action of L_0G , the connected component of LG of the identity.*

Proof. We show that $\tilde{\mathcal{L}}$ is constant on any orbit of L_0G , equivalently, for any path $\gamma(t)$ in $\tilde{\mathcal{C}}$ through $\gamma(0) = [x, \eta, [\tilde{x}]]$ along the L_0G -orbit, we need to prove

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \tilde{\mathcal{L}}(\gamma(t)) = 0.$$

We can assume that the tangent vector defined by $\gamma(t)$ is $(-\tilde{\xi}_x, \frac{\partial \xi}{\partial \theta} + [\eta, \xi])$ for $\xi \in L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$. Then the calculation in (3.9) implies that

$$\begin{aligned}
 &\left. \frac{\partial}{\partial t} \right|_{t=0} \tilde{\mathcal{L}}(\gamma(t)) \\
 &= \langle \nabla \tilde{\mathcal{L}}(x, \eta), (-\tilde{\xi}_x, \frac{\partial \xi}{\partial \theta} + [\eta, \xi]) \rangle \\
 &= \int_{S^1} \left(\omega(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x, -\tilde{\xi}_x) + \langle \mu(x), \frac{\partial \xi}{\partial \theta} + [\eta, \xi] \rangle \right) d\theta \\
 &= \int_{S^1} \left(\omega(\frac{\partial x}{\partial \theta}, -\tilde{\xi}_x) - \omega(\tilde{\eta}_x, \tilde{\xi}_x) + \langle \mu(x), \frac{\partial \xi}{\partial \theta} \rangle + \omega(\tilde{\eta}_x, \tilde{\xi}_x) \right) d\theta \\
 &= \int_{S^1} \left(\langle d\mu_x(\frac{\partial x}{\partial \theta}), \xi \rangle + \langle \mu(x), \frac{\partial \xi}{\partial \theta} \rangle \right) d\theta \\
 &= \int_{S^1} d\langle \mu(x), \xi \rangle = 0.
 \end{aligned}$$

Here we applied the equality: $\omega(\tilde{\eta}_x, \tilde{\xi}_x) = \langle \mu(x), [\eta, \xi] \rangle$. This completes the proof. \square

Given $(x, \eta) \in \text{Crit}(\mathcal{L})$ and $g \in LG$, by property (1) in Lemma 3.1, $g \cdot (x, \eta)$ is also a critical point. That means, the critical point set $\text{Crit}(\mathcal{L})$ is LG -invariant. Note that the based gauge group

$$\Omega G = \{g \in LG \mid g(1) = e, \text{ the identity element in } G\}$$

acts on \mathcal{C} freely. In the next lemma, we provide a description the critical point set modulo the group ΩG on the set theoretical level. For this purpose, we consider $C^\infty(S^1, \mathfrak{g})$ as the space of connections on the trivial bundle $S^1 \times G$, where we treat $\xi \in C^\infty(S^1, \mathfrak{g})$ as a \mathfrak{g} -valued 1-form $\xi d\theta$ on S^1 . Then there is a holonomy map

$$\text{Hol} : C^\infty(S^1, \mathfrak{g}) \longrightarrow G.$$

Note that $\text{Hol} : C^\infty(S^1, \mathfrak{g}) \longrightarrow G$ is the universal principal ΩG -bundle with ΩG -action on $C^\infty(S^1, \mathfrak{g})$ given by the gauge transformation.

Lemma 3.7. *Modulo the based gauge group ΩG , the holonomy map $\text{Hol} : \text{Crit}(\mathcal{C})/\Omega G \longrightarrow G$ defines a fibration over G whose fiber over $g \in G$ is $(\mu^{-1}(0))^g$, the g -fixed point set in $\mu^{-1}(0)$. That is, we have*

$$\text{Crit}(\mathcal{L})/\Omega G = \bigsqcup_{g \in G} (\mu^{-1}(0))^g.$$

Proof. Given $(x(\theta), \eta(\theta)) \in \text{Crit}(\mathcal{L})$, then $x(\theta) \in C^\infty(S^1, \mu^{-1}(0))$ and

$$\dot{x}(\theta) = -\tilde{\eta}_{x(\theta)}.$$

Solving the above ordinary differential equation over the interval $x : [0, 2\pi] \rightarrow X$ with an initial condition $x(0) = p \in \mu^{-1}(0)$, we get a unique solution. The condition of x be a loop in X is that η satisfies the condition

$$x(2\pi) = \text{Hol}(\eta) \cdot p = p.$$

Hence, we get

$$\text{Crit}(\mathcal{L}) \cong \{(p, \eta) \mid p \in \mu^{-1}(0), \eta \in C^\infty(S^1, \mathfrak{g}), \text{Hol}(\eta) \cdot p = p\}.$$

The action of ΩG on the right hand side is given by the gauge transformation on the second component. Note that the holonomy map $\text{Hol} : C^\infty(S^1, \mathfrak{g}) \rightarrow G$ is a principal ΩG -bundle. Any ΩG -orbit at η is determined by $\text{Hol}(\eta)$. So we get the first identification,

$$\text{Crit}(\mathcal{L})/\Omega G \cong \{(p, g) \mid p \in \mu^{-1}(0), g \in G_p\}.$$

Now it is easy to see that the holonomy map on $\{(p, g) \mid p \in \mu^{-1}(0), g \in G_p\}$ is just the projection to the second factor, whose fiber at g is $(\mu^{-1}(0))^g$. So the lemma is established. \square

Remark 3.8. Set-theoretically, the critical point set $\text{Crit}(\mathcal{L})/LG$ can be identified with

$$I[\mu^{-1}(0)/G] \cong (\mu^{-1}(0)/G) \sqcup \bigsqcup_{(e) \neq (g) \in \mathcal{C}(G)} (\mu^{-1}(0))^g/C(g),$$

the inertia groupoid arising from the action groupoid $[\mu^{-1}(0)/G] = \mu^{-1}(0) \rtimes G$. Here $\mathcal{C}(G)$ is the set of conjugacy class in G with a *choice* function $\mathcal{C}(G) \rightarrow G$ sending (g) to $g \in (g)$, and $C(g)$ is the centraliser of g in G .

- (1) If G acts on $\mu^{-1}(0)$ freely, then $\text{Crit}(\mathcal{L})/LG \cong \mu^{-1}(0)/G$ is the symplectic quotient (also called the reduced space) of (X, ω) .
- (2) If G -action on $\mu^{-1}(0)$ is only locally free, then $\text{Crit}(\mathcal{L})/LG$ is the inertia orbifold of the symplectic orbifold $\mathcal{X}_0 = [\mu^{-1}(0)/G]$.
- (3) If 0 is not a regular value of μ , then $\mu^{-1}(0)/G$ admits a symplectic orbifold stratified space, labelled by orbit types ([37]).

This remark suggests that the critical point set $\text{Crit}(\mathcal{L})/LG$ can be endowed with a symplectic orbifold structure when 0 is a regular value of μ and the functional is Bott-Morse type in this case. For this, we may investigate the Hessian operator of \mathcal{L} at $(x, \eta) \in \text{Crit}(\mathcal{L})$. In this paper, we only consider the case that 0 is a regular value of μ . The inertia orbifold of \mathcal{X}_0 is written as

$$I\mathcal{X}_0 = \bigsqcup_{(g)} \mathcal{X}_0^{(g)}$$

where (g) runs over the conjugacy class in G with non-empty fixed points in $\mu^{-1}(0)$. Note that for a non-trivial conjugacy class (g) , $\mathcal{X}_0^{(g)}$ is often called a twisted sector of \mathcal{X}_0 , which is diffeomorphic to the orbifold arising from the action of $C(g)$ on $\mu^{-1}(0)^g$ for a representative g in the conjugacy class (g) . Here $C(g)$ denotes the centralizer of g in G .

Proposition 3.9. *Assume that 0 is a regular value of μ . Let $(x, \eta) \in \text{Crit}(\mathcal{L})$, the Hessian operator of \mathcal{L} at (x, η)*

$$\text{Hess}_{(x, \eta)} : \Gamma_{C^\infty}(S^1, x^*TX \times \mathfrak{g}) \longrightarrow \Gamma_{C^\infty}(S^1, x^*TX \times \mathfrak{g})$$

is defined by $(v, \xi) \mapsto \left(J(\nabla_{\frac{\partial}{\partial \theta}} x^*v + \tilde{\xi} + \nabla_v \tilde{\eta}), d\mu_x(v) \right)$, which is a symmetric operator with respect to the inner product (3.6).

Proof. Given $(v_1, \xi_1), (v_2, \xi_2) \in T_{(x, \eta)}\mathcal{C} = \Gamma_{C^\infty}(S^1, x^*TX \times \mathfrak{g})$, the Hessian operator

$$\text{Hess}_{(x, \eta)} : T_{(x, \eta)}^{L^2}(\mathcal{C}_{1,p}) \rightarrow T_{(x, \eta)}^{L^2}(\mathcal{C}_{1,p})$$

is defined by the second Fréchet derivative

$$\langle (v_1, \xi_1), \text{Hess}_{(x, \eta)}(v_2, \xi_2) \rangle = D^2\mathcal{L}_{(x, \eta)}((v_1, \xi_1), (v_2, \xi_2)).$$

Denote by \bar{v}_1 the parallel transport along a path $\exp_x(sv_1)$.

$$\begin{aligned} & D^2\mathcal{L}_{(x, \eta)}((v_1, \xi_1), (v_2, \xi_2)) \\ &= \frac{d}{ds} \Big|_{s=0} ((D\mathcal{L})_{\exp_x(sv_2), \eta+s\xi_2}(\bar{v}_1, \xi_1)) \\ &= \frac{d}{ds} \Big|_{s=0} \left(\int_{S^1} \left(\omega_{\exp_x(sv_2)} \left(\frac{\partial(\exp_x(sv_2))}{\partial \theta} + \widetilde{(\eta + s\xi_2)}_{\exp_x(sv_2)}, \bar{v}_1 \right) + \langle \mu(\exp_x(sv_2)), \xi_1 \rangle \right) d\theta \right). \end{aligned}$$

Note that $\left. \frac{\partial \omega_{\exp_x(sv_2)}}{\partial s} \right|_{s=0} \left(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x, \cdot \right) = 0$, as $\frac{\partial x}{\partial \theta} + \tilde{\eta}_x = 0$. We can continue the above calculation as follows

$$\begin{aligned} &= \int_{S^1} \left(\omega(\nabla_{\frac{\partial}{\partial \theta}} x^* v_2, v_1) d\theta + \omega_x(\nabla_{v_2} \tilde{\eta} + \tilde{\xi}_2, v_1) d\theta + \langle d\mu_x(v_2), \xi_1 \rangle \right) \\ &= \int_{S^1} \left(\omega(\nabla_{\frac{\partial}{\partial \theta}} x^* v_2 + \tilde{\xi}_2 + \nabla_{v_2} \tilde{\eta}, v_1) + \langle d\mu_x(v_2), \xi_1 \rangle \right) d\theta \\ &= \int_{S^1} \left(\omega(\nabla_{\frac{\partial}{\partial \theta}} x^* v_2 + \tilde{\xi}_2 + \nabla_{v_2} \tilde{\eta}, v_1) + \langle d\mu_x(v_2), \xi_1 \rangle \right) d\theta. \end{aligned}$$

Here ∇ is the Levi-Civita covariant derivative associated to the Riemannian metric on X . So the Hessian operator $\text{Hess}_{(x,\eta)}$ is defined by

$$\text{Hess}_{(x,\eta)}(v, \xi) = \left(J(\nabla_{\frac{\partial}{\partial \theta}} x^* v + \tilde{\xi} + \nabla_v \tilde{\eta}), d\mu_x(v) \right).$$

It is a symmetric operator due to the following identity.

$$D^2 \mathcal{L}_{(x,\eta)}((v_1, \xi_1), (v_2, \xi_2)) = D^2 \mathcal{L}_{(x,\eta)}((v_2, \xi_2), (v_1, \xi_1)),$$

This identity can be proved using the following three identities.

- (1) $\int_{S^1} \left(\omega(\nabla_{\frac{\partial}{\partial \theta}} x^* v_1, v_2) - \omega(\nabla_{\frac{\partial}{\partial \theta}} x^* v_2, v_1) \right) d\theta = 0$.
- (2) $\int_{S^1} \left(\omega(\tilde{\xi}_2, v_1) + \langle d\mu_x(v_2), \xi_1 \rangle \right) d\theta = \int_{S^1} \left(\omega(\tilde{\xi}_2, v_1) + \langle \tilde{\xi}_1, v_2 \rangle \right) d\theta$, which is symmetric in (v_1, ξ_1) and (v_2, ξ_2) .
- (3) $\int_{S^1} \omega(\nabla_{v_2} \tilde{\eta}, v_1) d\theta = \int_{S^1} \omega(\nabla_{v_1} \tilde{\eta}, v_2) d\theta$.

□

We can choose a representative for any critical point in $\text{Crit}(\mathcal{L})$ according to its holonomy. If a critical point has a trivial holonomy, then using a based gauge transformation, it is gauge equivalent to a critical point of the form

$$(x, 0) \in \mu^{-1}(0) \times \mathfrak{g}.$$

If a critical point has a non-trivial holonomy $g = \exp(2\pi\eta)$ for $\eta \in \mathfrak{g}$, then it is gauge equivalent to a critical point of the form

$$(\exp(2\pi t\eta)x, \eta d\theta)$$

for $t \in [0, 2\pi]$ and $x \in (\mu^{-1}(0))^g$. As the Hessian operator is equivariant under the gauge transformation, it is often simpler to study the Hessian operator at critical points of the above special form.

Corollary 3.10. *The Hessian operator at the critical point $(x, 0) \in \mu^{-1}(0) \times \mathfrak{g}$ is given by*

$$\text{Hess}_{(x,0)}(v, \xi) = \left(J\left(\frac{\partial v}{\partial \theta} + \tilde{\xi}_x\right), d\mu_x(v) \right).$$

The Hessian operator at the critical point of the form $(x, \eta) = (\exp(2\pi\theta\eta)x_0, \eta)$ for $\theta \in [0, 2\pi]$ and $x_0 \in (\mu^{-1}(0))^g$ and $g = \exp(2\pi\eta)$ is given by

$$\text{Hess}_{(x,\eta)}(v, \xi) \mapsto \left(J(-L_{\tilde{\eta}}v + \tilde{\xi}_x), d\mu_x(v) \right).$$

Here $L_{\tilde{\eta}}v$ is the Lie derivative of v along the vector field $\tilde{\eta}$.

Proof. It is straightforward to check that the Hessian operator at a critical point of the form $(x, 0) \in \mu^{-1}(0) \times \mathfrak{g}$ is given by $(v, \xi) \mapsto \left(J\left(\frac{\partial v}{\partial \theta} + \tilde{\xi}_x\right), d\mu_x(v) \right)$.

At the critical point of the form $(x, \eta) = (\exp(2\pi\theta\eta)x_0, \eta)$, the vector field $\frac{\partial}{\partial \theta}$ along the loop $x = \exp(2\pi\theta\eta)x_0$ agrees with $-\tilde{\eta}$, then we get

$$\nabla_{\frac{\partial}{\partial \theta}} x^*v + \nabla_v \tilde{\eta} = -\nabla_{\tilde{\eta}} v + \nabla_v \tilde{\eta} = -L_{\tilde{\eta}}v.$$

Hence, the Hessian operator at this critical point is given by $(v, \xi) \mapsto \left(J(-L_{\tilde{\eta}}v + \tilde{\xi}_x), d\mu_x(v) \right)$. \square

Now we introduce the standard Banach completion of \mathcal{C} . This Banach set-up is also crucial for the Fredholm analysis of the gradient flowlines of \mathcal{L} , equivalently, the symplectic vortices on $S^1 \times \mathbb{R}$.

Consider the Banach manifold

$$\mathcal{C}_{1,p} = \{(x, \eta) \in W^{1,p}(S^1, X \times \mathfrak{g})\}.$$

Here $p \geq 2$, so (x, η) is a continuous map. For simplicity, one could just take $p = 2$. The tangent space of $\mathcal{C}_{1,p}$ at (x, η) is

$$T_{(x,\eta)}\mathcal{C}_{1,p} = \Gamma_{W^{1,p}}(S^1, x^*TX \times \mathfrak{g}),$$

consisting of $W^{1,p}$ -sections. The gauge group for this Banach manifold is the $W^{2,p}$ -loop group

$$\mathcal{G}_{2,p} = W^{2,p}(S^1, G)$$

acting on $\mathcal{C}_{1,p}$ in the way as in (3.1) for the smooth case. Denote by $\mathcal{G}_{2,p}^0$ the based $W^{2,p}$ -loop group. Then the action of $\mathcal{G}_{2,p}^0$ on $\mathcal{C}_{1,p}$ is free.

By the Sobolev embedding theorem, $T_{(x,\eta)}\mathcal{C}_{1,p}$ is contained in the L^2 -tangent space

$$T_{(x,\eta)}^{L^2}\mathcal{C}_{1,p} = \Gamma_{L^2}(S^1, x^*TX \times \mathfrak{g}),$$

the space of L^2 -section of the bundle $x^*TX \times \mathfrak{g}$ on which the L^2 -inner product (3.6) is well-defined and the L^2 -gradient $\nabla \mathcal{L}$ is a L^2 -tangent vector field on $\mathcal{C}_{1,p}$. Modulo $W^{2,p}$ gauge transformation, the equations (3.8) is a first order elliptic equation. By the standard elliptic regularity, we know that modulo gauge transformation, the critical point set $\text{Crit}(\mathcal{L})$ consists of smooth loops in $\mathcal{C}_{1,p}$. By the same argument, a solution to the L^2 gradient flow equation (3.10) of \mathcal{L} on $\mathcal{C}_{1,p}$ for

$$(x(t), \eta(t)) : [a, b] \longrightarrow \mathcal{C}_{1,p}$$

with a smooth initial boundary condition is gauge equivalent to a smooth symplectic vortex on $S^1 \times [a, b]$ in temporal gauge. In this sense, we say that the L^2 -gradient of \mathcal{L} and the L^2 -gradient flow lines are well-defined on $\mathcal{C}_{1,p}$.

Now we explain that the functional \mathcal{L} satisfies certain properties which are analogous to the Morse-Bott properties in the finite dimension.

Proposition 3.11. *Assume that 0 is a regular value of the moment map μ . Let (x, η) be a critical point of $\text{Crit}(\mathcal{L})$, then the Hessian operator $\text{Hess}_{(x, \eta)}$ of \mathcal{L} at (x, η)*

$$\text{Hess}_{(x, \eta)} : T_{(x, \eta)}^{L^2} \mathcal{C}_{1, p} \longrightarrow T_{(x, \eta)}^{L^2} \mathcal{C}_{1, p}$$

is an unbounded essentially self-adjoint operator whose spectrum is real, discrete and unbounded in both directions. Moreover, each non-zero eigenvalue has finite multiplicity, and the tangent space of the $\mathcal{G}_{2, p}$ -orbit through (x, η) , $T_{(x, \eta)}(\mathcal{G}_{2, p}(x, \eta))$ is contained in $\text{Ker}(\text{Hess}_{(x, \eta)})$, and its L^2 -orthogonal is finite dimensional.

Proof. Note that $\text{Crit}(\mathcal{L})$ is invariant under the gauge group $\mathcal{G}_{2, p}$ and the Hessian operator is $\mathcal{G}_{2, p}$ -equivariant. As 0 is a regular value of μ and X is compact, there are only finitely many elements of finite order in G with non-empty fixed points in $\mu^{-1}(0)$. By Lemma 3.7 and Remark 3.8, we know that any critical point is gauge equivalent to a critical point of the form

$$(x(\theta), \eta) = (\exp(\theta\eta)x_0, \eta)$$

for $x_0 \in \mu^{-1}(0)$ and $\eta \in \mathfrak{g}$ such that $\exp(2\pi\eta) \in G_{x_0}$ (a finite group). Therefore, we only need to establish the lemma for $(x(\theta), \eta) = (\exp(\theta\eta)x_0, \eta) \in \text{Crit}(\mathcal{L})$ for $\eta \in \mathfrak{g}$. Denote $g = \exp(2\pi\eta)$. Note that $(\exp(\theta\eta)x_0, \eta)$ is contained in a component of $\text{Crit}(\mathcal{L})$, whose quotient under the based gauge group $\mathcal{G}_{2, p}^0$ is diffeomorphic to

$$(\mu^{-1}(0))^g,$$

under the identification in Lemma 3.7.

In this case, we first prove the following L^2 -orthogonal decomposition

$$(3.11) \quad \text{Ker}(\text{Hess}_{(x, \eta)}) \cong T_{(x, \eta)}(\mathcal{G}_{2, p}^0 \cdot (x, \eta)) \oplus T_{x(0)}(\mu^{-1}(0))^g.$$

Here $T_{x(0)}(\mu^{-1}(0))$ is thought as a subspace of $\text{Ker}(\text{Hess}_{(x, \eta)})$ under the identification in Lemma 3.7. To check (3.11), let $(v, \xi) \in \text{Ker}(\text{Hess}_{(x, \eta)})$ be L^2 -orthogonal to $T_{(x, \eta_0)}(\mathcal{G}_{2, p} \cdot (x, \eta))$. Then (v, ξ) satisfies the following three equations.

$$(1) \quad \int_{S^1} \left(\omega(-\tilde{\zeta}_x, v) + \left\langle \frac{\partial \zeta}{\partial \theta} + [\eta, \zeta], \xi \right\rangle \right) d\theta = 0 \text{ for any } \zeta \in W^{2, p}(S^1, \mathfrak{g}).$$

$$(2) \quad -L_{\tilde{\eta}} v + \tilde{\xi}_{x(\theta)} = 0.$$

$$(3) \quad d\mu_{x(\theta)}(v) = 0 \Rightarrow v(\theta) \in T_{x(\theta)}(\mu^{-1}(0)).$$

Due to the identity $\omega(\tilde{\zeta}_x, v) = \langle d\mu_x(v), \zeta \rangle = 0$ for $v(\theta) \in T_{x(\theta)}(\mu^{-1}(0))$ and $\zeta \in W^{2, p}(S^1, \mathfrak{g})$, the first equation gives rise to

$$\frac{\partial \xi}{\partial \theta} + [\eta, \xi] = 0.$$

This equation admits a periodic solution ξ if and only if $[\eta, \xi] = 0$. Hence, ξ is a constant function taking value in the Lie algebra

$$\{\xi \in \mathfrak{g} \mid [\eta, \xi] = 0\}$$

of the centraliser of $g = \exp(2\pi\eta)$ in G . This implies that $\tilde{\xi}_{x(\theta)} \in T_{x(\theta)}(\mu^{-1}(0))^g$. Then any solution to the second equation is uniquely determined by an initial value $v(0) \in T_{x(0)}(\mu^{-1}(0))^g$.

The subspace of $T_{(x,\eta_0)}(\mathcal{C}_{1,p})$ which is L^2 -orthogonal to $T_{(x,\eta)}(\mathcal{G}_{2,p} \cdot (x, \eta))$ is given by

$$\{(v, \xi) | d\mu_x(v) + \frac{\partial v}{\partial \theta} + \tilde{\xi}_x = 0\},$$

on which the Hessian operator is a compact self-adjoint perturbation of a first order elliptic operator on S^1 . The remaining claims in the lemma follows from the standard elliptic theory on compact manifolds. \square

Denote by $\mathcal{C}_{1,p}^\#$ the submanifold of $\mathcal{C}_{1,p}$ consisting of elements with finite stabilisers under the gauge group $\mathcal{G}_{2,p}$. Then

$$\mathcal{B}_{1,p}^\# = \mathcal{C}_{1,p}^\# / \mathcal{G}_{2,p}$$

is a smooth Banach orbifold. Let $(x, \eta) \in \mathcal{C}_{1,p}^\#$ and let

$$\mathcal{G}_{(x,\eta)} = \{g \in \mathcal{G}_{2,p} | g \cdot (x, \eta) = (x, \eta)\}$$

be the stabiliser group of (x, η) , a finite group in $\mathcal{G}_{2,p}$. Then the tangent space at $\gamma = [x, \eta] \in \mathcal{B}_{1,p}^\#$ in orbifold sense is a $\mathcal{G}_{(x,\eta)}$ -invariant Banach space

$$\{(v, \xi) \in T_{(x,\eta)}\mathcal{C}_{1,p} | (v, \xi) \text{ is } L^2\text{-orthogonal to } T_{(x,\eta)}(\mathcal{G}_{2,p}(x, \eta))\}.$$

The action function \mathcal{L} descends locally to a circle-valued function on the Banach orbifold $\mathcal{B}_{1,p}^\#$. The L^2 -gradient vector field $\nabla \mathcal{L}$ defines an orbifold L^2 -gradient vector field on $\mathcal{B}_{1,p}^\#$. The following corollary follows from Lemma 3.7 and Proposition 3.11.

Corollary 3.12. *Assume that 0 is a regular value of the moment map μ , then the critical point set*

$$\text{Crit} = \text{Crit}(\mathcal{L}) / \mathcal{G}_{2,p} \subset \mathcal{B}_{1,p}^\#$$

is a smooth orbifold, diffeomorphic to the inertia orbifold of the symplectic reduction

$$\mathcal{X}_0 = [\mu^{-1}(0)/G].$$

Each component (called a twisted sector) is a finite dimensional suborbifold of $\mathcal{B}_{1,p}^\#$.

From now on, we assume that 0 is a regular value of the moment map μ . Let $[x, \eta] \in \text{Crit}(\mathcal{L}) / \mathcal{G}_{2,p}$. Then the Hessian operator at $[x, \eta] \in \text{Crit}(\mathcal{L}) / \mathcal{G}_{2,p}$ is an unbounded essentially self-adjoint Fredholm operator in the orbifold sense

$$\text{Hess}_{[x,\eta]} : T_{[x,\eta]}^{\text{orb}} \mathcal{B}_{1,p}^\# \longrightarrow T_{[x,\eta]}^{\text{orb}} \mathcal{B}_{1,p}^\#.$$

with discrete real spectrum (unbounded in both directions) of finite multiplicity. The kernel of $\text{Hess}_{[x,\eta]}$ is the orbifold tangent space of the critical orbifold at $[x, \eta]$ and the normal Hessian operator is nondegenerate. There is a uniformly lower bound on the absolute value of the non-zero eigenvalues of $\text{Hess}_{[x,\eta]}$ for $[x, \eta] \in \text{Crit}(\mathcal{L}) / \mathcal{G}_{2,p}$ as $\text{Crit}(\mathcal{L}) / \mathcal{G}_{2,p}$ is compact. We remark that the circle valued function \mathcal{L} on $\mathcal{B}_{1,p}^\#$ defines a closed 1-form on $\mathcal{B}_{1,p}^\#$ whose critical point set is of Morse-Bott type.

In the next lemma, we establish the inequality for $\tilde{\mathcal{L}}$ which is important in analysing gradient flow lines near any critical point.

Proposition 3.13. *For any x in a critical manifold $\text{Crit}(\tilde{\mathcal{L}}) \subset \tilde{\mathcal{C}}_{1,p}$, there exist a constant δ and a small $W^{1,p}$ ϵ -ball neighbourhood $B_\epsilon(x)$ of x in $\tilde{\mathcal{C}}_{1,p}$ such that*

$$\|\nabla \tilde{\mathcal{L}}(y)\|_{L^2}^2 \geq \delta |\tilde{\mathcal{L}}(y) - \tilde{\mathcal{L}}(x)|$$

for any $y \in B_\epsilon(x)$. Here ϵ and δ are independent of x (assuming that $\mu^{-1}(0)$ is compact).

We remark that though the above inequality is written in a small ϵ -ball of a critical point of $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}_{1,p}$, in fact the inequality still holds in a sufficiently small ϵ -ball of a critical point of \mathcal{L} on $\mathcal{C}_{1,p}$. This is due to the local diffeomorphism between $\tilde{\mathcal{C}}_{1,p}$ and $\mathcal{C}_{1,p}$. That is, the difference function $\mathcal{L}(y) - \mathcal{L}(x)$ makes sense for $y \in B_\epsilon(x)$ when ϵ is small.

Proof. By the gauge invariance, we only need to verify the inequality at critical points of the form

$$(\exp(\theta\eta)x_0, \eta)$$

for $x_0 \in \mu^{-1}(0)$ and $\eta \in \mathfrak{g}$ such that $\exp(2\pi\eta) \in G_{x_0}$ (a finite group). Assume that $\eta = 0$, then a small neighbourhood of $(x_0, 0)$ in $\tilde{\mathcal{C}}_{1,p}$ can be identified with a small ball in

$$T_{(x_0,0)}\mathcal{C}_{1,p} = W^{1,p}(S^1, T_{x_0}X \times \mathfrak{g})$$

centred at the origin with radius ϵ for a sufficiently small ϵ . Let $(u, \xi) \in W^{1,p}(S^1, T_{x_0}X \times \mathfrak{g})$ such that

$$\|(u, \xi)\|_{W^{1,p}} < \epsilon.$$

With respect to the canonical metric defined by $\omega(\cdot, J(\cdot))$, we have the following L^2 -orthogonal decomposition

$$T_{x_0}X \cong T_{x_0}\mu^{-1}(0) \oplus \nu_{x_0}$$

where $\nu_{x_0} = \{J(\tilde{\zeta}_{x_0})|\zeta \in \mathfrak{g}\}$. This decomposition provides a local coordinate of X at x_0 , denoted by (u_0, u_μ) . Under this coordinate, vector fields will be parallelly transported to the origin along the geodesic rays, can be treated as vectors in $T_{x_0}X$. Now we calculate

$$\|\nabla \tilde{\mathcal{L}}(u, \xi)\|_{L^2}^2 = \int_{S^1} \left(\left\| J \left(\frac{du}{d\theta} + \tilde{\xi}_u \right) \right\|^2 + \|\mu(u)\|^2 \right) d\theta$$

as follows. Write $\frac{du}{d\theta} = (\dot{u}_0, \dot{u}_\mu)$, we get the following estimates

$$\begin{aligned}
& \int_{S^1} \left\| J \left(\frac{du}{d\theta} + \tilde{\xi}_u \right) \right\|^2 d\theta \\
&= \int_{S^1} \|(\dot{u}_0 + \tilde{\xi}_{x_0}) + \dot{u}_\mu + (\tilde{\xi}_u - \tilde{\xi}_{x_0})\|^2 d\theta \\
&\geq \int_{S^1} \left(\|(\dot{u}_0 + \tilde{\xi}_{x_0})\|^2 + \|\dot{u}_\mu\|^2 - \|\tilde{\xi}_u - \tilde{\xi}_{x_0}\|^2 \right) d\theta \quad \text{as } \langle \dot{u}_0, \tilde{\xi}_{x_0} \rangle = \langle d\mu_{x_0}(\dot{u}_0), \xi \rangle = 0 \\
&\geq \int_{S^1} \left(\|\dot{u}_0\|^2 + \|\dot{u}_\mu\|^2 + \|\tilde{\xi}_{x_0}\|^2 - C\|u\|^2 \|\tilde{\xi}_{x_0}\|^2 \right) d\theta \quad \text{for some constant } C > 0 \\
&= \int_{S^1} \left(\|\dot{u}\|^2 + (1 - C\|u\|^2) \|\tilde{\xi}_{x_0}\|^2 \right) d\theta.
\end{aligned}$$

Here $\langle \dot{u}_0, \tilde{\xi}_{x_0} \rangle = \langle d\mu_{x_0}(\dot{u}_0), \xi \rangle = 0$ is applied in the calculation. Note that

$$\int_{S^1} \|\mu(u)\|^2 d\theta \geq \epsilon \|u_\mu\|_{L^\infty}^2,$$

for a sufficiently small ϵ . Hence, we obtain

$$\|\nabla \tilde{\mathcal{L}}(u, \xi)\|_{L^2}^2 \geq \|\dot{u}\|_{L^2}^2 + \epsilon \|u_\mu\|_{L^\infty}^2 + \epsilon \|\xi\|_{L^\infty}^2.$$

On the other hand, let $\tilde{u}(\theta, t) = tu(\theta)$ for $t \in [0, 1]$, then

$$\tilde{\mathcal{L}}(u, \xi) - \tilde{\mathcal{L}}(x_0, 0) = - \int_{S^1} \tilde{u}^* \omega + \int_{S^1} \langle \mu(u), \xi \rangle d\theta.$$

If $\int_{S^1} u_0 d\theta = 0 \in T_{x_0} \mu^{-1}(0)$, then

$$\left| \int_{S^1} \langle \mu(u), \xi \rangle d\theta \right| \leq C_1 (\|u_\mu\|_{L^\infty}^2 + \|\xi\|_{L^\infty}^2)$$

for some constant $c_1 > 0$. By a direct calculation, we know that

$$\left| \int_{S^1} \tilde{u}^* \omega \right| \leq C_2 \|u\|_{L^\infty} \int_{S^1} |\dot{u}| d\theta \leq C_3 \|u\|_{W^{1,2}},$$

for some constants C_2 and C_3 . Therefore, for a properly chosen $\delta > 0$ and sufficiently small ϵ , we have

$$\|\nabla \tilde{\mathcal{L}}(u, \xi)\|_{L^2}^2 \geq \delta |\tilde{\mathcal{L}}(u, \xi) - \tilde{\mathcal{L}}(x_0, 0)|.$$

If $\int_{S^1} u_0 d\theta \neq 0$, we can replace x_0 and $x'_0 = x_0 + \int_{S^1} u_0 d\theta$. Then we have

$$\tilde{\mathcal{L}}(x_0, 0) = \tilde{\mathcal{L}}(x'_0, 0).$$

The above calculation applied to $(x'_0, 0)$ implies

$$\left\| \nabla \tilde{\mathcal{L}}(u, \xi) \right\|_{L^2}^2 \geq \delta |\tilde{\mathcal{L}}(u, \xi) - \tilde{\mathcal{L}}(x'_0, 0)|.$$

So the inequality has been proved for any critical point which is gauge equivalent to $(x_0, 0)$. The above argument can be adapted for a critical point gauge equivalent to $(\exp(\theta\eta_0)x_0, \eta)$ by lifting technique: since $Hol(\eta)$ is of finite order, say k , then we consider a k covering $S^1 \rightarrow S^1$ and lift $(\exp(\theta\eta_0)x_0, \eta)$ to the first S^1 . Let $\tilde{\eta}$ be the lifting of η and then $Hol(\tilde{\eta})$ is trivial. Then we can use the above argument to get the estimate. \square

Remark 3.14. In the proof of Proposition 3.13, we use directly the explicit expression of $\tilde{\mathcal{L}}$ and $\nabla \tilde{\mathcal{L}}$ in the case that 0 is a regular value of μ . If 0 is irregular, then the argument in the proof does not apply. We can apply the Morse-Bott property of the functional $\tilde{\mathcal{L}}$ to establish a weaker inequality. In this paper, we only deal with the case when 0 is a regular value of μ , so we prefer to use the stronger inequality given by Proposition 3.13.

At the end of this subsection, we discuss the energy of gradient flow line. Let $\gamma = (\tilde{x}, \tilde{\eta}) : [a, b] \rightarrow \mathcal{C}_{1,p}$ be a path connecting (x_1, η_1) and (x_2, η_2) . Let $(\tilde{x}_1, \tilde{\eta}_1)$ be a path γ_1 connecting the based point to (x_1, η_1) ; then we set $(\tilde{x}_2, \tilde{\eta}_2)$ be the path $\gamma_2 = \gamma_1 \# \gamma$. As in Remark 3.3, we treat $(\tilde{x}_1, \tilde{\eta}_1)$ and $(\tilde{x}_2, \tilde{\eta}_2)$ as elements in $\tilde{\mathcal{C}}$.

Lemma 3.15. *Suppose that $\gamma = \gamma(t) : [a, b] \rightarrow \mathcal{C}_{1,p}$ is a gradient flowline of \mathcal{L} . Then the following quatity are equal:*

- (1) $\tilde{\mathcal{L}}(\tilde{x}_1, \tilde{\eta}_1) - \tilde{\mathcal{L}}(\tilde{x}_2, \tilde{\eta}_2)$
- (2) $-\int_{S^1 \times [a,b]} \tilde{x}^* \omega + \int_{S^1} (\langle \mu(x_1), \eta_1 \rangle - \langle \mu(x_2), \eta_2 \rangle) d\theta$
- (3) *the Yang-Mills-Higgs energy $E(\tilde{x}, \tilde{\eta})$;*
- (4) $\int_a^b \left\| \frac{\partial \gamma(t)}{\partial t} \right\|_{L^2}^2 dt.$

Proof. Recall that a path $\gamma = (x, \eta)$ is a gradient flow line of $\nabla \mathcal{L}$ if it satisfies the equations

$$\frac{\partial}{\partial t} (x(t), \eta(t)) = - \left(J \left(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x \right), \mu(x) \right).$$

Since $(\tilde{x}, \tilde{\eta})$ solves the symplectic vortex equation,

$$E(\tilde{x}, \tilde{\eta}) = - \int_{S^1 \times [a,b]} \tilde{x}^* \omega + d \langle \mu(\tilde{x}), \tilde{\eta} \rangle.$$

This implies that (1)=(2)=(3). Now we show that (1)=(4).

$$\tilde{\mathcal{L}}(\tilde{x}_2, \tilde{\eta}_2) - \tilde{\mathcal{L}}(\tilde{x}_1, \tilde{\eta}_1) = \int_a^b \frac{d}{dt} \mathcal{L}(\tilde{x}(t), \tilde{\eta}(t)) dt = \int_a^b \langle \nabla \mathcal{L}, \frac{d\gamma}{dt} \rangle dt = \int_a^b \left\| \frac{\partial \gamma(t)}{\partial t} \right\|_{L^2}^2 dt.$$

□

3.2. Asymptotic behaviour of finite energy symplectic vortices on a cylinder.

In this subsection, we establish the existence of a limit point for any gradient flow line

$$\gamma : [0, \infty) \rightarrow \mathcal{C}_{1,p}$$

with finite energy $E(\gamma)$. Then by Lemma 3.15, we have

$$(3.12) \quad \int_0^\infty \left\| \frac{\partial \gamma(t)}{\partial t} \right\|_{L^2}^2 dt = \int_0^\infty \|\nabla \mathcal{L}(\gamma(t))\|_{L^2}^2 dt = E(\gamma) < \infty.$$

Theorem 3.16. *Let $\gamma : [0, \infty) \rightarrow \mathcal{C}_{1,p}$ be a gradient flow line of \mathcal{L} with finite energy. Then there exists a unique critical point $(x_\infty, \eta_\infty) \in \text{Crit}(\mathcal{L})$ and constants $\delta, C > 0$ such that the L^2 -distance*

$$\text{dist}_{L^2}(\gamma(T), (x_\infty, \eta_\infty)) \leq C e^{-\delta T}$$

for any sufficient large T .

Proof. Step 1. For any sequence $\{\gamma(t_i) \mid \lim_{i \rightarrow \infty} t_i = \infty\}$, we show that there exists a convergent subsequence, still denoted by $\{\gamma(t_i)\}$, such that up to gauge transformations in $\mathcal{G}_{2,p}$, the sequence $\{\gamma(t_i)\}$ converges to a critical point y_∞ of \mathcal{L} in the C^∞ -topology.

Let $(u_i, A_i) = \gamma(t_i)$ be the symplectic vortex on $S^1 \times [-1, 1]$ in temporal gauge, obtained from $\gamma : [t_i - 1, t_i + 1] \rightarrow \mathcal{C}_{1,p}$. Then we have

$$\lim_{i \rightarrow \infty} E(u_i, A_i) = 0,$$

where the energy $E(u_i, A_i)$ agrees with the Yang-Mills-Higgs energy

$$E(u_i, A_i) = \int_{S^1 \times [-1, 1]} \frac{1}{2} (|d_{A_i} u_i|^2 + |F_{A_i}|^2 + |\mu(u_i)|^2) d\theta dt.$$

Applying the standard regularity result and Uhlenbeck compactness, see Theorem 3.2 in [12], there exists a sequence of $W^{2,p}$ -gauge transformations g_i of $P_{S^1} \times [-1, 1]$ such that the sequence

$$g_i \cdot (u_i, A_i)$$

has a C^∞ -convergent subsequence. Let (u_∞, A_∞) be the limit, then (u_∞, A_∞) satisfies the following equations

$$(3.13) \quad F_{A_\infty} = 0, \quad d_{A_\infty} u_\infty = 0, \quad \mu(u_\infty) = 0.$$

We can find a smooth gauge transformation h of $P_{S^1} \times [-1, 1]$ such that $h \cdot (u_\infty, A_\infty)$ is in temporal gauge. So we can write

$$h \cdot (u_\infty, A_\infty) = (x_\infty(\theta, t), \eta_\infty(\theta, t) d\theta)$$

as a path in $\mathcal{C}_{1,p}$. Then the equations (3.13) become

$$\begin{cases} \frac{\partial \eta_\infty(\theta, t)}{\partial t} = 0, \frac{\partial x_\infty(\theta, t)}{\partial t} = 0 \\ \frac{\partial x_\infty}{\partial \theta} + \tilde{\eta}_\infty(x_\infty) = 0, \mu(x_\infty) = 0. \end{cases}$$

These equations imply that $(x_\infty, \eta_\infty) = h \cdot (u_\infty, A_\infty) \in \text{Crit}(\mathcal{L})$ and

$$\lim_{i \rightarrow \infty} (hg_i) \cdot (u_i, A_i) = (x_\infty, \eta_\infty)$$

in the C^∞ -topology. Hence, up to gauge transformations in $\mathcal{G}_{2,p}$, the subsequence $\{\gamma(t_i)\}$ converges to a critical point (x_∞, η_∞) of \mathcal{L} in the C^∞ -topology. We denote it by y_∞ .

Step 2. Set $\gamma^i = g_i \gamma$. We claim that there exists i such that $\gamma^i(t) \in B_\epsilon(y_\infty)$ for $t \geq t_i$. Here ϵ is the constant given in Proposition 3.13.

If not, for each i there exists $s_i > t_i$ such that the path $\gamma^i(t), t_i \leq t \leq s_i$ locates in $B_\epsilon(y_\infty)$ and $\gamma^i(s_i) \in \partial B_\epsilon(y_\infty)$. Then

$$\begin{aligned} \text{dist}_{L^2}(\gamma^i(t_i), \gamma^i(s_i)) &\leq \int_{t_i}^{s_i} \left\| \frac{\partial \gamma^i(t)}{\partial t} \right\|_{L^2} dt = \int_{t_i}^{s_i} \|\nabla(\mathcal{L}(\gamma^i(t)) - \mathcal{L}(y_\infty))\|_{L^2} dt \\ &\leq -2\delta^{-1} \int_{t_i}^{s_i} \frac{d}{dt} ((\mathcal{L}(\gamma^i(t)) - \mathcal{L}(y_\infty))^{1/2}) \\ &\leq 2\delta^{-1} ((\mathcal{L}(\gamma^i(t_i)) - \mathcal{L}(y_\infty))^{1/2} - (\mathcal{L}(\gamma^i(s_i)) - \mathcal{L}(y_\infty))^{1/2}). \end{aligned}$$

As $i \rightarrow \infty$, this goes to 0. On the other hand, by Step 1, there exists h_i such that $h_i \gamma^i(s_i)$ uniform converges to some critical point y'_∞ . Since $\gamma^i(s_i) \in B_\epsilon(y_\infty)$ and $h_i \gamma^i(s_i)$ uniformly converges to y'_∞ , h_i is uniformly bounded at least in $C^{1,\alpha}$ for some $\alpha > 0$. This means that there exists a subsequence of h_i that converges. We may relabel the sequence and assume that h_i converges to h . We conclude that $\gamma^i(s_i)$ converges to $h^{-1}y'_\infty$. Therefore,

$$\text{dist}_{L^2}(\gamma^i(t_i), \gamma^i(s_i)) \rightarrow \text{dist}_{L^2}(y_\infty, h^{-1}y'_\infty) = 0.$$

This implies that $y_\infty = h^{-1}y'_\infty$. However, $\gamma^i(s_i)$ is on the boundary of the ball $B_\epsilon(y_\infty)$, this is impossible. The contradiction implies Step 2.

Step 3: From Step 2, suppose that $\gamma^i(t)$ locates in $B_\epsilon(y_\infty)$ when t large. Reset y_∞ to be $g_i^{-1}y_\infty$. Then we may assume that $\gamma(t)$ locates in $B_\epsilon(y_\infty)$ when t large. Now we show that

$$\text{dist}_{L^2}(\gamma(t), y_\infty) \leq Ce^{-\delta t}$$

for t large.

We can assume that for $t > T_0$, $\gamma(t) \in B_\epsilon(y_\infty)$ so that the crucial inequality in Proposition 3.13 can be applied to get

$$\frac{d(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty))^{1/2}}{dt} = -\|\nabla(\mathcal{L}(\gamma(t)))\|_{L^2} \leq -\delta(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty))^{1/2}.$$

Hence, for any $t > T_0$, we have

$$\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty) \leq e^{-\delta(t-T_0)}(\mathcal{L}(\gamma(T_0)) - \mathcal{L}(y_\infty)).$$

That is, for $t > T_0$

$$\text{dist}_{L^2}(\gamma(t), y_\infty) \leq 2ce^{-\delta(t-T_0)}(\mathcal{L}(\gamma(T_0)) - \mathcal{L}(y_\infty))^{1/2}.$$

Take $C = 2ce^{T_0}(\mathcal{L}(\gamma(T_0)) - \mathcal{L}(y_\infty))^{1/2}$, we get the exponential decay estimate for $\text{dist}_{L^2}(\gamma(t), y_\infty)$. \square

Remark 3.17. By a similar calculation as the proof, one can establish the following exponential decay for a finite energy gradient flow line $\gamma : [0, \infty) \rightarrow \mathcal{C}_{1,p}$. That is, there exist constants $\delta, C > 0$ such that

$$\int_T^\infty \|\nabla \mathcal{L}(\gamma(t))\|_{L^2}^2 dt \leq Ce^{-\delta T}$$

for a sufficiently large T . Moreover, let y_∞ be the limit of $\gamma(t)$ at infinity, by a gauge transformation, we may assume that $y_\infty \in \text{Crit}$, then for any $k \in \mathbb{N}$, there exist $C, \delta > 0$ such that

$$(3.14) \quad |\nabla^k \gamma(t)| \leq Ce^{-\delta t}$$

for t sufficiently large. To get the above point-wise estimate, we apply the elliptic regularity to the symplectic vortex $\gamma|_{[T-2, T+2] \times S^1}$ for a sufficiently large T to get a C^k -estimate

$$\|g \cdot \gamma\|_{C^k} \leq C.$$

for some constant $C > 0$ and any $k \in \mathbb{N}$. Write $\gamma = (\alpha, u)$, then the curvature F_α and $\mu(u)$ are gauge invariant and hence bounded. Then (3.14) follows from applying the standard elliptic

estimates to the gradient flow equations. We also remark that the decay rate δ can be chosen such that δ is smaller than the minimum absolute value of non-zero eigenvalues of the Hessian operator of \mathcal{L} at y_∞ .

4. L^2 -MODULI SPACE OF SYMPLECTIC VORTICES ON A CYLINDRICAL RIEMANN SURFACE

In this section, we consider the symplectic vortices of finite energy on a Riemann surface Σ with cylindrical end. For simplicity, Σ is assumed to have just one end, isometrically diffeomorphic to a half cylinder $S^1 \times [0, \infty)$. Let K be a compact set of Σ such that $\Sigma \setminus K$ is isometrically diffeomorphic to $S^1 \times (1, \infty)$ with the flat metric $d\theta^2 + dt^2$. Let P be a principal G -bundle over Σ and $\mathcal{N}_\Sigma(X, P)$ be the moduli space of symplectic vortices with finite energy associated to P and a closed Hamiltonian manifold (X, ω) . It is the space of gauge equivalence classes of

$$(A, u) \in \mathcal{A}(P) \times C_G^\infty(P, X)$$

satisfying the symplectic vortex equations (2.1) and with the property that the Yang-Mills-Higgs energy (Cf. (2.4)) is finite. The main result of this section is to prove the continuity for the asymptotic limit map established in Section 3

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow \text{Crit},$$

where Crit is the critical point set modulo the gauge transformations. We remark that Crit is diffeomorphic to the inertia orbifold $I\mathcal{X}_0$ associated to the reduced symplectic orbifold $\mathcal{X}_0 = [\mu^{-1}(0)/G]$, as we assume that 0 is a regular value of the moment map μ .

To study the moduli space $\mathcal{N}_\Sigma(X, P)$, we consider the $W^{1,p}$ -space

$$\tilde{\mathcal{B}}_{W_{\text{loc}}^{1,p}(\Sigma)} = \mathcal{A}_{W_{\text{loc}}^{1,p}(\Sigma)} \times W_{\text{loc}, G}^{1,p}(P, X)$$

for $p \geq 2$. Then by the elliptic regularity, $\mathcal{N}_\Sigma(X, P)$ is the space of solutions to the symplectic vortex equations (2.1) for $(A, u) \in \tilde{\mathcal{B}}_{W_{\text{loc}}^{1,p}(\Sigma)}$ such that

$$E(A, u) = \int_\Sigma \frac{1}{2}(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2) \nu_\Sigma < \infty$$

modulo the action of the group $\mathcal{G}_{W_{\text{loc}}^{2,p}(\Sigma)}$ of all $W_{\text{loc}}^{2,p}$ gauge transformations.

Proposition 4.1. *Let Σ be a Riemann surface Σ with one cylindrical end, P be a principal G -bundle over Σ and $\mathcal{N}_\Sigma(X, P)$ be the moduli space of symplectic vortices with finite energy associated to P and a closed Hamiltonian manifold (X, ω) . Then the asymptotic limit of symplectic vortices in $\mathcal{N}_\Sigma(X, P)$ define a continuous map*

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow \text{Crit} \cong I\mathcal{X}_0.$$

Proof. Let $[(u, A)] \in \mathcal{N}_\Sigma(X, P)$ and $\partial_\infty([(u, A)]) = y_\infty$ correspond to $[x_0] \in (\mu^{-1}(0))^g/C(g)$. Fix an open neighbourhood V of $[x_0]$ in $(\mu^{-1}(0))^g/C(g)$, a twisted sector in $I\mathcal{X}_0$. We need to find an open neighbourhood $U \subset \mathcal{N}_\Sigma(X, P)$ of $[(u, A)]$ such that

$$\partial_\infty(U) \subset V.$$

Let $\tilde{\mathcal{V}}$ be a $\mathcal{G}_{2,p}(S^1)$ -invariant open neighbourhood of $(\exp(2\pi\theta\eta_0)x_0, \eta_0)$ in $\mathcal{C}_{1,p}$ such that $\tilde{\mathcal{V}} \cap \text{Crit}(\mathcal{L})$ is mapped to a subset of V under the identification

$$\text{Crit}(\mathcal{L})/\mathcal{G}_{2,p}(S^1) \cong I\mathcal{X}_0.$$

Denote by $\tilde{\mathcal{U}}$ the solutions $(u, A) \in \tilde{\mathcal{B}}_{W_{\text{loc}}^{1,p}(\Sigma)}$ to the symplectic vortex equations with finite energy such that for sufficiently large T , the restriction of (u, A) to $S^1 \times [T, \infty)$ is gauge equivalent to an element in $\tilde{\mathcal{V}}$. Then $U = \tilde{\mathcal{U}}/\mathcal{G}_{W^{2,p}(\Sigma)}$ is an open neighbourhood of $[(u, A)]$ in $\mathcal{N}_\Sigma(X, P)$ and $\partial_\infty(U) \subset V$. \square

4.1. Fredholm theory for L^2 -moduli space of symplectic vortices. To understand the moduli space $\mathcal{N}_\Sigma(X, P)$, we need to introduce the weighted Sobolev space for the fiber of the asymptotic limit map

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow \text{Crit} \cong I\mathcal{X}_0.$$

Any symplectic vortex $[(u, A)] \in \mathcal{N}_\Sigma(X, P)$ decays exponentially to its asymptotic limit $\partial_\infty([(u, A)])$ at a rate $\delta > 0$ for some δ such that δ is smaller than the minimum absolute value of non-zero eigenvalues of the Hessian operator of \mathcal{L} at $\partial_\infty([(u, A)])$. Note that Crit is compact, so we can choose a constant δ such that $[(u, A)] \in \partial_\infty^{-1}(y_\infty)$ decays exponentially to y_∞ at the rate δ for any $y_\infty \in \text{Crit}$. We fix such a δ throughout this section.

Fix a smooth function $\beta : \Sigma \rightarrow [0, \infty)$ such that the follow conditions hold:

- (1) On $S^1 \times [1, \infty)$, β is the coordinate function on the cylinder.
- (2) $\beta = 0$ on $\Sigma \setminus \{S^1 \times [0, \infty)\}$.
- (3) $\beta|_{S^1 \times [0, 1]}$ is an increasing function.

The weighted $W^{k,p}$ -norm on a compact support section ξ of an Euclidean vector bundle V over Σ with a covariant derivative ∇ is defined by

$$\|\xi\|_{W_\delta^{k,p}} = \left(\int_\Sigma e^{\delta\beta} (|\xi|^p + |\nabla(\xi)|^p + \cdots + |\nabla^p(\xi)|^p) d\nu_\Sigma \right)^{1/p}.$$

We denote $W_\delta^{k,p}(\Sigma, E)$ the completion of all compact support sections of E with respect to the weighted $W^{k,p}$ -norm, which is also called the Banach space of $W_\delta^{k,p}$ -sections of E . When $k = 0$, we simply denote by $L_\delta^p(\Sigma, E)$ the $W^{0,p}$ -sections of E .

Let $(A_\infty, u_\infty) \in \text{Crit}(\mathcal{L})$. Let by pulling back, we get a (A_0, u_0) on cylinder end $S^1 \times [1, \infty)$, namely it is constant in $t \in [1, \infty)$ and agrees with (A_∞, u_∞) .

Define $\tilde{\mathcal{B}}_\delta(A_\infty, u_\infty)$ be the subspace of $\tilde{\mathcal{B}}_{W_{\text{loc}}^{1,p}}$ consisting of elements (A, u) with the following property:

- (1) $A - A_0 \in W_\delta^{1,p}(S^1 \times [1, \infty), \Lambda^1 \otimes P^{ad})$,
- (2) there exists a sufficiently large T depending on (A, u) such that $u|_{S^1 \times [T, \infty)} = \exp_{u_\infty}(v)$ for $v \in W_\delta^{1,p}(S^1 \times [T, \infty), u_\infty^*TX)$.

Then $\tilde{\mathcal{B}}_\delta(A_\infty, u_\infty)$ is a Banach manifold whose tangent space at (A, u) is given by

$$T_{(A,u)}\tilde{\mathcal{B}}_\delta(A_\infty, u_\infty) = W_\delta^{1,p}(\Sigma, \Lambda^1 \otimes P^{ad} \oplus u^*TX).$$

Define

$$\tilde{\mathcal{B}}_\delta = \bigcup_{(A_\infty, u_\infty)} \tilde{\mathcal{B}}_\delta(A_\infty, u_\infty) \rightarrow \text{Crit}(\mathcal{L}).$$

This is a smooth family of Banach manifold.

The gauge group \mathcal{G}_δ in this setting is defined as follows. For each $g_\infty \in \mathcal{G}(P_{S^1})$, we construct a g_0 on $S^1 \times [1, \infty)$ by pulling back g_∞ . Then we define $\mathcal{G}_\delta(g_\infty)$ similarly using $W_\delta^{2,p}$ -norm. Set \mathcal{G}_δ be the union of $\mathcal{G}_\delta(g_\infty)$. Set

$$\mathcal{B}_\delta = \tilde{\mathcal{B}}_\delta / \mathcal{G}_\delta.$$

The symplectic vortex equation (2.1) defines a smooth \mathcal{G}_δ -invariant section S of the \mathcal{G}_δ -equivariant Banach bundle $\tilde{\mathcal{E}}_\delta$ whose fiber at (A, u) is given by

$$(\tilde{\mathcal{E}}_\delta(A_\infty, u_\infty))_{(A, u)} = L_\delta^p(\Sigma, \Lambda^{0,1} \otimes u^* T^{\text{vert}} Y \oplus P^{ad})$$

where $Y = P \times_G X$, where $(A, u) \in \tilde{\mathcal{B}}_\delta(A_\infty, u_\infty)$. The moduli space

$$\mathcal{N}_\Sigma(X, P) = S^{-1}(0) / \mathcal{G}_\delta.$$

We explain $\partial_\infty^{-1}(y_\infty)$. Let (A_∞, u_∞) be a representative of y_∞ . Note that by our assumption, (A_∞, u_∞) has only a finite group as its isotropy group, denoted by G_∞ . Then we may define $\mathcal{G}_\delta(A_\infty, u_\infty)$ is a disjoint union of $\mathcal{G}_\delta(g)$ for $g \in G_\infty$. The connected component of the identity is the Banach Lie group whose Banach Lie algebra is $W_\delta^{2,p}(\Sigma, P^{ad})$.

The symplectic vortex equation (2.1) defines a smooth $\mathcal{G}_\delta(A_\infty, u_\infty)$ -invariant section $S_{(A_\infty, u_\infty)}$ of the $\mathcal{G}_\delta(A_\infty, u_\infty)$ -equivariant Banach bundle $\tilde{\mathcal{E}}_\delta$ whose fiber at (A, u) is given by

$$(\tilde{\mathcal{E}}_\delta(A_\infty, u_\infty))_{(A, u)} = L_\delta^p(\Sigma, \Lambda^{0,1} \otimes u^* T^{\text{vert}} Y \oplus P^{ad})$$

where $Y = P \times_G X$. The deformation complex associated to a symplectic vortex $(A, u) \in \tilde{\mathcal{B}}_\delta(A_\infty, u_\infty)$ is given by

$$W_\delta^{2,p}(\Sigma, P^{ad}) \xrightarrow{L_{A,u}} W_\delta^{1,p}(\Sigma, \Lambda^1 \otimes P^{ad} \oplus u^* TX) \xrightarrow{\mathcal{D}_{A,u}} L_\delta^p(\Sigma, \Lambda^{0,1} \otimes u^* T^{\text{vert}} Y \oplus P^{ad}),$$

which is elliptic in the sense that the cohomology groups are finite dimensional. The proof of this statement is quite standard nowadays so we omit it here. See the books [13], [28] and [38]. This ensures that the quotient of

$$(\tilde{\mathcal{B}}_\delta(A_\infty, u_\infty), \tilde{\mathcal{E}}_\delta(A_\infty, u_\infty), S_{(A_\infty, u_\infty)})$$

the gauge group $\mathcal{G}_\delta(A_\infty, u_\infty)$ is a Fredholm system. By the exponential decay result for symplectic vortices in $\mathcal{N}_\Sigma(X, P)$, we know that it is indeed a Fredholm system for the fiber of the asymptotic limit map at (A_∞, u_∞) . The index of this Fredholm system can be computed by the Atiyah-Patodi-Singer index formula for elliptic differential operators on manifolds with cylindrical end. The theorem below relates this index with the expected dimension for orbifold symplectic vortices.

Note that the formal dimension of the moduli space $\mathcal{N}_\Sigma(X, P)$ at a point $[(A, u)]$ is given by the index of the operator $\mathcal{D}_{A,u} \oplus L_{A,u}^*$ associated to the $(-\delta)$ -weighted deformation complex. For the purpose of the calculation, we can replace the operator for (A, u) on Σ to by a suitable

operator on an associated orbifold Riemann surface. We construct this replacement as follows. Assume that

$$\partial_\infty(A, u) = (A_\infty, u_\infty) = (\xi d\theta, \exp(\theta\xi) \cdot x_\infty)$$

where $g = \exp(2\pi\xi)$ has order m and $x_\infty \in (\mu^{-1}(0))^g$. Note that $[(A_\infty, u_\infty)]$ determines an element in $(\mu^{-1}(0))^g/C(g) \subset \text{Crit}$, a twisted sector of the reduced orbifold $\mathcal{X}_0 = \mu^{-1}(0)/G$. We can identify the cylinder $S^1 \times [0, \infty)$ with a unit disc $\mathbb{D}^* = \mathbb{D} - \{0\}$ in \mathbb{C} using the coordinate change $(i\theta, t) \mapsto e^{-(t+i\theta)}$. Then the cylindrical surface Σ become a punctured Riemann surface. Denote this punctured Riemann surface by Σ^* . Let P^* be the corresponding principal G -bundle over Σ^* .

As the connection A_∞ has a non-trivial holonomy, A does not extend to a connection P^* . Consider the degree m covering map $\phi : \mathbb{D} - \{0\} \rightarrow \mathbb{D} - \{0\}$ defined by $re^{i\theta} \mapsto (re^{i\theta})^m$. Then $\phi^*(A_\infty, u_\infty)$, as a rotation invariant symplectic vortex on \mathbb{D}^* is gauge equivalent to $(0, x_\infty)$ which is a \mathbb{Z}_m -invariant symplectic vortex on \mathbb{D} , or extends to a constant symplectic vortex on the orbifold $[\mathbb{D}/\mathbb{Z}_m]$ associated to a trivialized principal G -bundle.

Let Σ_{orbi} be the orbifold Riemann surface whose underlying topological space is the closure of Σ^* . The punctured point p is treated as a singular point locally modelled on $(\mathbb{D}, \mathbb{Z}_m)$ with the action of \mathbb{Z}_m on \mathbb{D} is generated by the multiplication $(e^{2\pi i/m}, z) \mapsto e^{2\pi i/m}z$. Then the above discussion implies that (A, u) can be replaced by a pair (\tilde{A}, \tilde{u}) on the orbifold Riemann surface associated to $(P_{\text{orbi}}, X, \omega)$ such that the chosen trivialization at the orbifold point p is specified by a based point \tilde{p} in the fiber of P_{orbi} , and

$$\tilde{u}(\tilde{p}) = x_\infty \in (\mu^{-1}(0))^g,$$

where $g = \exp(2\pi\xi)$ and x_∞ are determined by the asymptotic value as above. Hence, \tilde{u} gives rise to a degree 2 equivariant homology class in $H_2(X_G, \mathbb{Z})$. For simplicity, we still denote this class by $[u_G]$ which is called the **homology class of** (A, u) . Then by a direct calculation, the energy of (A, u) is

$$E(A, u) = \langle [\omega - \mu], [u_G] \rangle.$$

Fix an equivariant homology class $B \in H_2(X_G, \mathbb{Z})$ such that $\langle [\omega - \mu], B \rangle > 0$. Let $\mathcal{N}_\Sigma(X, P, B)$ be the moduli space of symplectic vortices on Σ associated to (P, X) with the homology class B . Then $\mathcal{N}_\Sigma(X, P, B) \subset \mathcal{N}_\Sigma(X, P)$ and the asymptotic limit map in Proposition 4.1 defines a continuous map on $\mathcal{N}_\Sigma(X, P, B)$

$$\partial_\infty : \mathcal{N}_\Sigma(X, P, B) \rightarrow \text{Crit}.$$

To calculate the expected dimension of components in $\mathcal{N}_\Sigma(X, P, B)$ over $(\mu^{-1}(0))^g/C(g)$, we introduce a degree shift as in [10]. We first define the degree shift of an element g in G of order m acting on \mathbb{C}^n . Let the complex eigenvalues of g on \mathbb{C}^n be

$$e^{2\pi i m_1/m}, e^{2\pi i m_2/m}, \dots, e^{2\pi i m_n/m}$$

for an n -tuple of integers (m_1, m_2, \dots, m_n) with $0 \leq m_j < m$ for $j = 1, 2, \dots, n$. Then the degree shift of an element g on \mathbb{C}^n , denoted by $\iota(g, \mathbb{C}^n)$, is given by

$$\iota(g, \mathbb{C}^n) = \sum_{j=1}^n \frac{m_j}{m}.$$

From the definition, we have

$$\iota(g, \mathbb{C}^n) + \iota(g^{-1}, \mathbb{C}^n) = n.$$

For the orbifold $\mathcal{X}_0 = [\mu^{-1}(0)/G]$, if $g \in G$ has a non-empty fixed point set $(\mu^{-1}(0))^g$, then the Chen-Ruan degree shift of g on \mathcal{X}_0 , denoted by $\iota_{CR}(g, \mathcal{X}_0)$ at $x \in (\mu^{-1}(0))^g$, is defined to be

$$\iota_{CR}(g, \mathcal{X}_0) = \iota(g, T_{[x]}(\mathcal{X}_0)).$$

For a twisted sector $\mathcal{X}_0^{(g)}$ of \mathcal{X}_0 , the corresponding degree shift as in [10] is defined to

$$\iota_{CR}(\mathcal{X}_0^{(g)}, \mathcal{X}_0) = \iota_{CR}(g, \mathcal{X}_0)$$

for any g such that $\mathcal{X}_0^{(g)}$ is diffeomorphic to the orbifold defined by the action of $C(g)$ on the fixed point manifold $(\mu^{-1}(0))^g$.

Theorem 4.2. *Let $\mathcal{N}_\Sigma(X, P, B; (g))$ be the subset of $\mathcal{N}_\Sigma(X, P, B)$ consisting of symplectic vortices $[(A, u)]$ such that*

$$\partial_\infty(A, u) \in \mathcal{X}_0^{(g)} \subset \text{Crit}$$

Then $\mathcal{N}_\Sigma(X, P, B; (g))$ admits a Fredholm system with its virtual dimension given by

$$2\langle c_1^G(TX), B \rangle + 2(n - \dim G)(1 - g_\Sigma) - 2\iota_{CR}(\mathcal{X}_0^{(g)}, \mathcal{X}_0)$$

where g_Σ is the genus of the Riemann surface Σ .

Proof. With the Fredholm set-up and the orbifold model discussed above, we only need to calculate the index of the linearisation operator for the symplectic vortex (\tilde{A}, \tilde{u}) on the orbifold Riemann surface $\Sigma_{orb} = (|\Sigma_{orb}|, (p, m))$, modulo based gauge transformations. Note that the remaining gauge transformations consist of constant ones taking values in $C(g)$, the centraliser of g in G , as $\tilde{u}(\tilde{p}) \in (\mu^{-1}(0))^g$.

The underlying Fredholm operator is the a compact perturbation of the direct sum of the operator $(-d_A^*, *d_A)$

$$\Omega_\delta^1(\Sigma, P^{ad}) \rightarrow \Omega_\delta^0(\Sigma, P^{ad}) \oplus \Omega_\delta^0(\Sigma, P^{ad})$$

in the original cylindrical model, with its index given by $-\dim G(1 - 2g_\Sigma)$, and the Cauchy-Riemann operator $\bar{\partial}_{\tilde{A}, \tilde{u}}$ on the orbifold Σ_{orb} with values in the complex vector bundle $\tilde{u}^*T^{\text{vert}}Y$. Hence, the virtual dimension is given by

$$(4.1) \quad \text{Index} \bar{\partial}_{\tilde{A}, \tilde{u}} - \dim G(1 - 2g_\Sigma) - \dim C(g).$$

By the orbifold index theorem, we have

$$(4.2) \quad \text{Index} \bar{\partial}_{\tilde{A}, \tilde{u}} = 2\langle c_1(u^*T^{\text{vert}}Y), [|\Sigma_{orb}|] \rangle + 2n(1 - g_\Sigma) - 2\iota_{CR}(g, T_{\tilde{u}(\tilde{p})}X).$$

By the definition of $c_1^G(u^*T^{\text{vert}}Y)$ and $[u_G]$, we have

$$\langle c_1(u^*T^{\text{vert}}Y), [[\Sigma_{\text{orbi}}]] \rangle = \langle c_1^G(TX), B \rangle.$$

To calculate the degree shift for the g -action on $T_{\tilde{u}(\tilde{p})}X$, we apply the following decomposition

$$T_{\tilde{u}(\tilde{p})}X \cong \mathfrak{g} \oplus \mathfrak{g}^* \oplus T_{[x_\infty]}\mathcal{X}_0.$$

Here the actions of g on \mathfrak{g} and \mathfrak{g}^* are adjoint to each other and the zero eigenspace of the g -action on \mathfrak{g} is the Lie algebra of $C(g)$. By the definition of degree shift, this implies that

$$\begin{aligned} (4.3) \quad & 2\iota_{CR}(g, T_{\tilde{u}(\tilde{p})}X) \\ &= 2\iota_{CR}(g, \mathcal{X}_0) + 2\dim_{\mathbb{C}} G/C(g) \\ &= 2\iota_{CR}(g, \mathcal{X}_0) + \dim_{\mathbb{R}} G/C(g). \end{aligned}$$

Put these formula (4.1), (4.2) and (4.3) together, we get the virtual dimension as claimed in the theorem. \square

4.2. L^2 -moduli space of symplectic vortices on punctured Riemann surface.

Let $C = (\Sigma, p_1, \dots, p_k)$ be a Riemann surface with k marked points. We assume that C is stable, i.e, $2 - 2g(\Sigma) - k < 0$ where g_Σ is the genus of the Riemann surface Σ . It is well known that there is a canonical hyperbolic metric on the punctured Riemann surface $\Sigma \setminus \{p_1, \dots, p_k\}$. This hyperbolic metric provides a disjoint union of horodisks $D(p_i)$ rounded at each punctured point p_i . We may deform the metric on the disc such that the metric becomes a cylinder end metric. For simplicity, we use the same notation Σ for this Riemann surface with k cylindrical ends. Denote the metric and the corresponding volume form by ρ_Σ and ν_Σ respectively.

Let P be a principal G -bundle over Σ . Let $\mathcal{N}_\Sigma(X, P)$ be the moduli space of symplectic vortices with finite energy on Σ associated to P and a $2n$ -dimensional Hamiltonian G -space (X, ω) . Then $\mathcal{N}_\Sigma(X, P)$ is the space of gauge equivalence classes of solutions to the symplectic vortex equations (2.1) for

$$(A, u) \in \tilde{\mathcal{B}}_{W_{\text{loc}}^{1,p}(\Sigma)} = \mathcal{A}_{W_{\text{loc}}^{1,p}(\Sigma)} \times W_{\text{loc},G}^{1,p}(P, X)$$

such that

$$E(A, u) = \int_{\Sigma} \frac{1}{2}(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2) \nu_\Sigma < \infty.$$

Then the asymptotic limit map

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow (\text{Crit})^k$$

is continuous. Let δ be a positive real number which is smaller than the minimum absolute value of eigenvalues of the Hessian operators of \mathcal{L} along the compact critical manifold Crit , then $[u, A] \in \mathcal{N}_\Sigma(X, P)$ decays exponentially to its asymptotic limit along each end. Moreover, the energy function on $\mathcal{N}_{\Sigma_p}(X, P)$ takes values in a discrete set

$$\{\langle [\omega - \mu], B \rangle | B \in H_2^G(X, \mathbb{Z})\}.$$

Fix an equivariant homology class $B \in H_2^G(X, \mathbb{Z})$ such that $\langle [\omega - \mu], B \rangle > 0$. Let $\mathcal{N}_\Sigma(X, P, B)$ be the moduli space of symplectic vortices on Σ associated to (P, X) with the homology class

B . We remark that the homology class of (u, A) is defined by the associated orbifold model as in the previous section for one cylindrical end cases.

Then the Fredholm analysis for a one cylindrical end case in the previous section can be adapted to establish the following theorem.

Theorem 4.3. *Let $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})$ be the subset of $\mathcal{N}_\Sigma(X, P, B)$ consisting of symplectic vortices $[(A, u)]$ such that*

$$\partial_\infty(A, u) \in (\mathcal{X}_0^{(g_1)} \times \cdots \times \mathcal{X}_0^{(g_k)}) \subset (\text{Crit})^k$$

Then $\mathcal{N}_\Sigma(X, P, B; \{g_i\}_{i=1, \dots, k})$ admits an orbifold Fredholm system with its virtual dimension given by

$$2\langle c_1^G(TX), B \rangle + 2(n - \dim G)(1 - g_\Sigma) - 2 \sum_{i=1}^k \iota_{CR}(\mathcal{X}_0^{(g_i)}, \mathcal{X}_0)$$

where g_Σ is the genus of the Riemann surface Σ .

5. COMPACTNESS OF L^2 -MODULI SPACE OF SYMPLECTIC VORTICES

In this section, we establish a compactness result for the underlying topological space of the moduli space $\mathcal{N}_\Sigma(X, P, B)$ of symplectic vortices on a Riemann surface Σ with k cylindrical ends. We assume that $k > 0$. By reversing the orientation on S^1 if necessarily, we can assume that all these ends are modelled on $S^1 \times [0, \infty)$.

Given an orbifold topological space \mathcal{N} , the underlying topological space (also called the coarse space of \mathcal{N}) will be denoted by $|\mathcal{N}|$. We will provide a compactification of the coarse moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ by adding certain limiting data consisting of bubbling off J -holomorphic spheres in (X, ω, J) as in the Gromov-Witten theory and bubbled chains of symplectic vortices on cylinders. When X is Kähler, the compactness theorem for the L^2 -moduli spaces of symplectic vortices on a Riemann surface with cylindrical end have been studied in [39].

To describe the limiting data for a sequence of symplectic vortices on Σ , we introduce an index set for the topological type of the domain. Let g be the topological genus of Σ and $B \in H_2^G(X, \mathbb{Z})$ such that $\langle [\omega - \mu], B \rangle > 0$. Recall that a tree is a connected graph without any closed cycle of edges.

Definition 5.1. A web of stable weighted trees of the type $(\Sigma; B)$ is a finite disjoint union of trees

$$\Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$$

consisting of a principal tree Γ_0 with ordered k tails and a collection of chains (ordered sequences) of trees

$$\Gamma_i = \bigsqcup_{j=1}^{m_i} T_i(j)$$

for each tail $i = 1, \dots, k$, together with the following additional conditions.

- (1) The principal tree Γ_0 has a distinguished vertex (called the principal root) with a weight (g, B_0) and ordered k tails labelled by $\{1, 2, \dots, k\}$. Here $B_0 \in H_2^G(X, \mathbb{Z})$ satisfies the positivity condition

$$\langle [\omega - \mu], B_0 \rangle \geq 0.$$

- (2) For the i -th tail in Γ_0 , there is a chain of trees of length m_i

$$\Gamma_i = T_i(1) \sqcup T_i(2) \sqcup \dots \sqcup T_i(m_i)$$

such that, if non-empty, each $T_i(j)$ has a distinguished vertex (called a branch root) with a weight given by a class $B_{i,j} \in H_2^G(X, \mathbb{Z})$ such that

$$\langle [\omega - \mu], B_{i,j} \rangle \geq 0.$$

If $B_{i,j} = 0$, the tree $T_i(j)$ is non-trivial in the sense that the branch root is not the only vertex.

- (3) Any undistinguished vertex v in Γ has its weight given by a class $B_v \in H_2(X, \mathbb{Z})$. such that

$$\langle [\omega - \mu], B_v \rangle \geq 0.$$

If $B_v = 0$, the number of edges at v is at least 3, two of which have non-zero weights.

- (4) Under the natural homomorphism $H_2(X, \mathbb{Z}) \rightarrow H_2^G(X, \mathbb{Z})$,

$$(5.1) \quad B_0 + \sum_{i=1}^k \sum_{j=1}^{m_i} B_{i,j} + \sum_{i=0}^m \sum_{v \in V(\Gamma_i)} B_v = B.$$

Here $V(\Gamma_i)$ is the set of undistinguished vertices in Γ_i .

The equivalence between two webs of stable weighted trees can be defined in a usual sense. Denote by $\mathcal{S}_{\Sigma; B}$ be the set of equivalence classes of webs of stable weighted trees of the type $(\Sigma; B)$.

Given two element $[\Gamma]$ and $[\Gamma']$ in $\mathcal{S}_{\Sigma; B}$, we say $[\Gamma] \prec [\Gamma']$ if any representative Γ' in $[\Gamma']$ can be obtained, up to equivalence, from any representative Γ in $[\Gamma]$ by performing finitely many steps of the following three operations.

- (1) Contracting an edge connecting two undistinguished vertices, say v_1 and v_2 , in a tree to obtain a new vertex with a new weight $B_{v_1} + B_{v_2}$.
- (2) Identifying two branch roots of adjacent trees in a chain Γ_i to get a chain of trees of length $m_i - 1$ with a weight given by the sum of the two assigned weights.
- (3) Identifying the principal root in Γ with a first branch root in a chain, say Γ_i , such that the new main root is endowed with a new weight $B_0 + B_{i,1}$ and the i -th chain becomes $T_i(2) \sqcup \dots \sqcup T_i(m_i)$.

Lemma 5.2. $(\mathcal{S}_{\Sigma; B}, \prec)$ is a partially ordered finite set.

Proof. It is easy to see that the order \prec is a partial order. By the condition (5.1), we see that there are only finitely many collections of

$$\{(B_0, B_{i,j}, B_v) | \langle [\omega - \mu], B_{i,j} \rangle > 0, \langle [\omega - \mu], B_v \rangle > 0\}.$$

The stability conditions for branch roots or undistinguished vertices with zero weight implies that there are only finitely many possibilities. This ensures that $\mathcal{S}_{\Sigma;B}$ is a finite set. \square

Given an element $\Gamma = \sqcup_{i=0}^k \Gamma_i$ in $\mathcal{S}_{\Sigma;B}$, we can associate a bubbled Riemann surface of genus g and k cylindrical ends, and a collection of chains of bubbled cylinders as follows. Associated to Γ_0 , we assign a bubbled Riemann surface Σ_0 which is the nodal Riemann surface obtained by attaching trees of \mathbb{CP}^1 's to Σ . Associated to an i -th chain of trees $\Gamma_i = \sqcup_{j=1}^{m_i} T_i(j)$ we assign a chain of bubbled cylinders

$$C_i = \{C_i(1), \dots, C_i(m_i)\}$$

where each $C_i(j)$ is a nodal cylinder with trees of \mathbb{CP}^1 's attached according to the tree.

Now we construct a moduli space of stable symplectic vortices with the domain curve being the bubbled Riemann surface Σ_0 or one of the bubbled cylinders in $\{C_i(j) | i = 1, \dots, k; j = 1, \dots, m_i\}$ as follows.

Let $\tilde{\Gamma}_0$ be the new weighted graph obtained by severing all edges in Γ_0 which are attached to the root. Assume that Γ_0 has l_0 trees attached to the root. Then $\tilde{\Gamma}_0$ consists a single vertex (the root) with l_0 half-edges and k ordered tails. The remaining part of Γ_0 , denoted by $\hat{\Gamma}_0$, becomes a disjoint union of l_0 trees, each of which has a half-edge attached one particular vertex (the adjacent vertex to the root). Equivalently,

$$\Gamma_0 = (\tilde{\Gamma}_0 \sqcup \hat{\Gamma}_0) / \sim$$

where the equivalence relation is given by the identification of l_0 -tuple half-edges in $\tilde{\Gamma}_0$ with the l_0 -tuple half-edges in $\hat{\Gamma}_0$.

Denote by $\mathcal{N}_{\tilde{\Gamma}_0}$ by the moduli space of symplectic vortices of homology class B_0 over Σ with l_0 marked points and k_0 cylindrical ends. Then there is a continuous map

$$\tilde{ev}_0 : \mathcal{N}_{\tilde{\Gamma}_0} \longrightarrow X^{l_0}$$

given by the evaluations at the l_0 marked points. Moreover, there is a continuous asymptotic limit map along each of the k cylindrical ends

$$\partial_0 : \mathcal{N}_{\tilde{\Gamma}_0} \longrightarrow (\text{Crit})^k.$$

Associated to $\hat{\Gamma}_0$, as a disjoint union of l_0 trees, there is a moduli space of the Gromov-Witten moduli space of unparametrized stable pseudo-holomorphic spheres with l_0 -marked points and the weighted dual graph given by $\hat{\Gamma}_0$, see Chapter 5 in [30]. We denote this moduli space by $\mathcal{M}_{\hat{\Gamma}_0}^{GW}$. Then there is a continuous map

$$\hat{ev}_0 : \mathcal{M}_{\hat{\Gamma}_0}^{GW} \longrightarrow X^{l_0}$$

given by the evaluations at the l_0 marked points. The moduli space of bubbled symplectic vortices of type Γ_0 , denoted by \mathcal{N}_{Γ_0} , is defined to be the orbifold topological space *generated* by the fiber product

$$\mathcal{N}_{\tilde{\Gamma}_0} \times_{X^{l_0}} \mathcal{M}_{\hat{\Gamma}_0}^{GW}$$

with respect to the maps $\tilde{e}v_0$ and $\hat{e}v_0$. Then the coarse moduli space $|\mathcal{N}_{\Gamma_0}|$ inherits a continuous asymptotic limit map

$$\partial_{\Gamma_0} : |\mathcal{N}_{\Gamma_0}| \longrightarrow (\text{Crit})^k.$$

Remark 5.3. We remark that there is an ambiguity here with regarding the orbifold structure on \mathcal{N}_{Γ_0} . A proper way to make this precise is to employ the language of proper étale groupoids to describe the spaces of objects and arrows on $\mathcal{N}_{\hat{\Gamma}_0} \times_{X^{l_0}} \mathcal{M}_{\hat{\Gamma}_0}^{GW}$, and then add further arrows to include all equivalences relations to get an orbifold structure on \mathcal{N}_{Γ_0} . As we are dealing with the compactification of the coarse moduli space, there is no ambiguity for the coarse space $|\mathcal{N}_{\Gamma_0}|$. We will return to this issue when we discuss weak Freholm systems for these moduli spaces in [8] and [9].

Similarly, for the i -th chain of trees $\Gamma_i = \bigsqcup_{j=1}^{m_i} T_i(j)$, we define a chain of moduli spaces of stable symplectic vortices of type Γ_i as follows. Associated to the tree $T_i(j)$, we excise the branch root away to get a graph consisting of a single vertex with $l_{i,j}$ half-edges and $l_{i,j}$ trees with one half-edge for each tree. Let $\tilde{T}_i(j)$ and $\hat{T}_i(j)$ be these two graphs respectively. Denote by $\mathcal{N}_{i,j}$ be the moduli space of symplectic vortices of homology class $B_{i,j}$ over the cylinder $C_i(j) \cong S^1 \times \mathbb{R}$ with $l_{i,j}$ -marked points. Then there are a continuous evaluation map

$$\tilde{e}v_{i,j} : \mathcal{N}_{i,j} \longrightarrow X^{l_{i,j}}$$

and continuous asymptotic value maps

$$\partial_{i,j}^{\pm} : \mathcal{N}_{i,j} \longrightarrow \text{Crit}$$

associated to the two ends at $\pm\infty$ respectively. Denote by $\mathcal{M}_{i,j}^{GW}$ the Gromov-Witten moduli space of unparametrized stable pseudo-holomorphic spheres with l_0 -marked points and the weighted dual graph given by $\hat{\Gamma}_0$. Note that $\mathcal{M}_{i,j}^{GW}$ is equipped with a continuous evaluation map

$$\tilde{e}v_{i,j} : \mathcal{M}_{i,j}^{GW} \longrightarrow X^{l_{i,j}}.$$

Then by adding all arrows to the fiber product

$$\mathcal{N}_{i,j} \times_{X^{l_{i,j}}} \mathcal{M}_{i,j}^{GW},$$

we get the moduli space $\hat{\mathcal{N}}_{T_i(j)}$ of stable symplectic vortices of type $T_i(j)$. In particular, $|\hat{\mathcal{N}}_{T_i(j)}|$ inherits continuous asymptotic limit maps

$$(5.2) \quad \hat{\partial}_{T_i(j)}^{\pm} : |\hat{\mathcal{N}}_{T_i(j)}| \longrightarrow \text{Crit}$$

along the two ends. Note that the group of rotations and translations $S^1 \times \mathbb{R}$ on the cylinder induces a free action of $S^1 \times \mathbb{R}$ on the moduli space $\hat{\mathcal{N}}_{T_i(j)}$ which preserves the asymptotic limit maps $\hat{\partial}_{T_i(j)}^{\pm}$ invar. We quotient the moduli space $\hat{\mathcal{N}}_{T_i(j)}$ by the group $\mathbb{R} \times S^1$, and denote the resulting moduli space by

$$\mathcal{N}_{T_i(j)} = \hat{\mathcal{N}}_{T_i(j)} / (\mathbb{R} \times S^1).$$

The induced asymptotic limit maps on the coarse moduli space is denoted by

$$\partial_{T_i(j)}^{\pm} : |\mathcal{N}_{T_i(j)}| \longrightarrow \text{Crit}.$$

By taking the consecutive fiber products with respect to maps $\partial_{T_i(j)}^+$ and $\partial_{T_i(j+1)}^-$ for $j = 1, \dots, m_i$, we get the coarse moduli spaces of chains of stable symplectic vortices of type Γ_i , that is,

$$|\mathcal{N}_{\Gamma_i}| = |\mathcal{N}_{T_i(1)}| \times_{\text{Crit}} |\mathcal{N}_{T_i(2)}| \times_{\text{Crit}} \cdots \times_{\text{Crit}} |\mathcal{N}_{T_i(m_i)}|.$$

Then there are two asymptotic limit maps given by $\partial_{T_i(1)}^-$ and $\partial_{T_i(m_i)}^+$, simply denoted by ∂_i^- and ∂_i^+ ,

$$\partial_i^\pm : |\mathcal{N}_{\Gamma_i}| \longrightarrow \text{Crit}.$$

Definition 5.4. Given $\Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$ a web of stable weighted trees in $\mathcal{S}_{\Sigma;B}$, the coarse moduli space of stable symplectic vortices of type Γ , denoted by $|\mathcal{N}_\Gamma|$, is defined to be the fiber product

$$|\mathcal{N}_\Gamma| = |\mathcal{N}_{\Gamma_0}| \times_{(\text{Crit})^k} \prod_{i=1}^k |\mathcal{N}_{\Gamma_i}|,$$

where $\prod_{i=1}^k |\mathcal{N}_{\Gamma_i}| = |\mathcal{N}_{\Gamma_1}| \times |\mathcal{N}_{\Gamma_2}| \times \cdots \times |\mathcal{N}_{\Gamma_k}|$, and the fiber product is defined via the maps $\partial_{\Gamma_0} : |\mathcal{N}_{\Gamma_0}| \rightarrow (\text{Crit})^k$ and

$$\prod_{i=1}^k \partial_i^- : \prod_{i=1}^k |\mathcal{N}_{\Gamma_i}| \rightarrow (\text{Crit})^k.$$

There exists a continuous map

$$\partial_\Gamma : |\mathcal{N}_\Gamma| \longrightarrow (\text{Crit})^k$$

given by $\prod_{i=1}^k \partial_i^+$.

For any k -tuple $((g_1), \dots, (g_k))$ conjugacy classes in G such that each representative g_i in (g_i) has a non-empty fixed point set in $\mu^{-1}(0)$, then we define

$$|\mathcal{N}_\Gamma((g_1), \dots, (g_k))| = \partial_\Gamma^{-1} \left(|\mathcal{X}_0^{(g_1)}| \times \cdots \times |\mathcal{X}_0^{(g_k)}| \right).$$

Now we can state the compactness theorem for the coarse L^2 -moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ of symplectic vortices on Σ .

Theorem 5.5. *Let Σ be a Riemann surface of genus g with k -cylindrical ends. The coarse L^2 -moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ can be compactified to a stratified topological space*

$$|\overline{\mathcal{N}}_\Sigma(X, P, B)| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma|$$

such that the top stratum is $|\mathcal{N}_\Sigma(X, P, B)|$. Moreover, the coarse moduli space

$$|\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})|$$

of the moduli space $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})$ in Theorem 4.3 can be compactified to a stratified topological space

$$|\overline{\mathcal{N}}_\Sigma(X, P, B; \{(g_i)\}_{i=1, \dots, k})| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma((g_1), \dots, (g_k))|.$$

Proof. For simplicity, we assume that the Riemann surface Σ has only one outgoing cylindrical end, that is, diffeomorphic to $S^1 \times [0, \infty)$. The proof for the general case is essentially the same. Under this assumption, any web of stable weighted trees of the type $(\Sigma; B)$ has only one chain of trees denoted by $\{T(1), T(2), \dots, T(m)\}$.

Given any sequence $[A_i, u_i] \in \mathcal{N}_\Sigma(X, P, B)$, we shall show that there exists a subsequence with a limiting datum in \mathcal{N}_Γ for some $\Gamma \in \mathcal{S}_{\Sigma; B}$. The strategy to this claim is quite standard now, for example see [16], [13] and [33].

Note that the energy function on this sequence

$$E(A_i, u_i) = \int_\Sigma \frac{1}{2} (|d_{A_i} u_i|^2 + |F_{A_i}|^2 + |\mu \circ u_i|^2) \nu_\Sigma$$

is constant given by $\langle [\omega - \mu], B \rangle$. For any non-constant pseudo-holomorphic map from a closed Riemann surface, the energy is bounded from below by a positive number

$$\min\{\langle [\omega], \beta \rangle \mid \beta \in H_2(X, \mathbb{Z}), \langle [\omega], \beta \rangle > 0\},$$

which is greater than the minimal energy of non-trivial symplectic vortices on Σ associated to (P, X, ω)

$$\hbar = \min\{\langle [\omega - \mu], \beta \rangle \mid \beta \in H_2^G(X, \mathbb{Z}), \langle [\omega - \mu], \beta \rangle > 0\},$$

Step 1. (Convergence for the sequence with bounded derivative) Without loss of generality, we suppose that $\{(A_i, u_i)\}$ is a sequence of symplectic vortices in $\mathcal{N}_\Sigma(X, P, B)$ with a uniform bound

$$\|d_{A_i} u_i\|_{L^\infty} < C$$

for a constant C . Then there exists a sequence of gauge transformations $\{g_i\}$ such that $\{g_i \cdot (A_i, u_i)\}$ has a C^∞ convergent subsequence.

This claim follows from Theorem 3.2 in [12].

Step 2. (Bubbling phenomenon at interior points) Assume that the sequence $\|d_{A_i} u_i\|_{L^\infty}$ is unbounded over a compact set in Σ , then the rescaling technics in the proof of Theorem 3.4 in [12] can be applied here to get the standard pseudo-holomorphic sphere at the point in Σ where a sphere is attached to Σ .

Hence, combining Steps 1-2, we know that there may exist a subset of finite points, say $\{q_1, \dots, q_{l_0}\}$, of Σ such that for any compact set $Z \subset \Sigma' = \Sigma - \{q_1, \dots, q_{l_0}\}$, there exists a subsequence of (A_i, u_i) and gauge transformation g_i such that $g_i(A_i, u_i)$ uniformly converge in Z . As Z exhausts Σ' , we get a symplectic vortex (A_∞, u_∞) on Σ' . By the removable singularity theorem, this symplectic vortex (A_∞, u_∞) can be defined on Σ .

Moreover, at each point q_j , we get a bubble tree of holomorphic sphere attached to q_i . As in the Gromov-Witten theory, there is no energy lost when the bubbling phenomenon happens at interior points. This gives rise to a principal tree Γ_0 in a web of stable weighted trees in $\mathcal{S}_{\Sigma; B}$.

Step 3. (Bubbling phenomenon at the infinite end) Assuming that for a sufficiently large T , the sequence $\{(A_i, u_i)\}$ converges to (A_∞, u_∞) on $\Sigma - (S^1 \times [T, \infty))$, where (A_∞, u_∞) is of the type Γ_0 , a principal tree in Definition 5.1. Now we study the sequence over the cylindrical end.

We may further assume that the Yang-Mills-Higgs energy

$$\int_{S^1 \times [T, \infty)} \frac{1}{2} (|d_{A_i} u_i|^2 + |F_{A_i}|^2 + |\mu \circ u_i|^2)$$

is greater than the minimum energy \hbar defined as above. Otherwise, the limit of the sequence is in \mathcal{N}_{Γ_0} .

We replace the sequence $\{(A_i, u_i)|_{(S^1 \times [T, \infty))}\}$ by their translations to the left by $\{\delta_i\}$ such that the Yang-Mills-Higgs energy of the translate for (A_i, u_i) over $[T - \delta_i, 0]$ is $\hbar/4$. Then $\delta_i \rightarrow \infty$ as $i \rightarrow \infty$. Applying the above standard convergence theorem to the translated sequence with and without the bounded derivative condition on $\|d_{A_i} u_i\|_{L^\infty}$, there exists a subsequence which converges to a bubbled symplectic vertex (A'_∞, u'_∞) on any compact subset of $S^1 \times \mathbb{R}$. This gives rise to a stable symplectic vortex of type $\Gamma(1)$, where $\Gamma(1)$ is a tree with a branch root as in Definition 5.1.

Step 5. (No energy loss in between) Now we show that there is no energy loss on the connecting neck between (A'_∞, u'_∞) and (A_∞, u_∞) . Equivalently, associated to the subsequence (still denoted by $\{(A_i, u_i)\}$, for each i , there exist

$$N_i < N_i + K_i < N'_i - K_i < N'_i$$

such that N_i, K_i and $N'_i - N_i - 2K_i \rightarrow \infty$ as $i \rightarrow \infty$, and under the temporal gauge,

- (1) the sequence (A_i, u_i) on $S^1 \times [N_i, N_i + K_i]$ coneverages to (A_∞, u_∞) on any compact set after translation;
- (2) (A_i, u_i) on $S^1 \times [N'_i - K_i, N'_i]$ coneverages to (A'_∞, u'_∞) on any compact set after translation.

We shall show that the Yang-Mills-Higgs energy of (A_i, u_i) on $S^1 \times [N_i + K_i, N'_i - K_i]$ tends to 0 as $i \rightarrow \infty$.

Let y_∞ and $y'_{-\infty}$ be the limit of (A_∞, u_∞) as $t \rightarrow \infty$ and (A'_∞, u'_∞) as $t \rightarrow -\infty$ respectively. Let $\bar{y}'_{-\infty}$ be the pair obtained from $y'_{-\infty}$ by reversing the orientation of S^1 . Suppose that

$$y_\infty = (a, \alpha), \quad \bar{y}'_{-\infty} = (b, \beta).$$

Then $(A_i(t), u_i(t)), t \in [N_i, N_i + K_i]$ is arbitrary close to y_∞ and $(A_i(t'), u_i(t')), t' \in [N'_i - K_i, N'_i]$ is arbitrary close to $\bar{y}'_{-\infty}$ as $i \rightarrow \infty$.

We claim that $\tilde{\mathcal{L}}(y_\infty) = \tilde{\mathcal{L}}(\bar{y}'_{-\infty})$. Otherwise, the difference would be larger than \hbar . However the Yang-Mills-Higgs energy of (A_i, u_i) on $[N_i, N'_i]$ is less than $\hbar/2$. This is impossible.

Now we explain the Yang-Mills-Higgs energy of (A_i, u_i) at $[N_i + t, N'_i - t]$ decays exponentially with respect to t . We normalize the band by translation such that $[N_i + K_i, N'_i - K_i]$ becomes $[-d, d]$ where $d = \frac{N'_i - N_i}{2} - K_i$.

Denote the Yang-Mills-Higgs energy of $y_i = (A_i, u_i)$ on $S^1 \times [-t, t]$ by

$$E_i(t) = \int_{S^1 \times [-t, t]} \frac{1}{2} (|d_{A_i} u_i|^2 + |F_{A_i}|^2 + |\mu \circ u_i|^2)$$

for $0 \leq t \leq d$. Then

$$\frac{dE_i(t)}{dt} = \|\nabla \tilde{\mathcal{L}}_{y_i(t)}\|^2 + \|\nabla \tilde{\mathcal{L}}_{y_i(-t)}\|^2$$

Replace $\tilde{\mathcal{L}}$ by $\tilde{\mathcal{L}} - \tilde{\mathcal{L}}(y_\infty)$. Then by the crucial inequality (Proposition 3.13), we obtain the following differential inequality

$$(5.3) \quad \frac{dE_i(t)}{dt} \geq \delta(|\tilde{\mathcal{L}}(y_i(t))| + |\tilde{\mathcal{L}}(y_i(-t))|) \geq \delta E_i(t).$$

Here we use the fact that

$$E_i(t) = |\tilde{\mathcal{L}}(y_i(t)) - \tilde{\mathcal{L}}(y_i(-t))|.$$

Then the differential inequality (5.3) implies

$$e^{-\delta t} E_i(t) \leq e^{-\delta d} E_i(d).$$

Apply to our case, this implies

$$(5.4) \quad E(A_i, u_i)|_{[N_i+K_i, N'_i-K_i]} \leq e^{-\delta K_i} E(A_i, u_i)|_{[N_i, N'_i]}.$$

As $K_i \rightarrow \infty$, the Yang-Mills-Higgs energy goes to 0.

This ensures that

$$\partial_{\Gamma_0}([A_\infty, u_\infty]) = \partial_{\Gamma_1}^{-1}([A'_\infty, u'_\infty]) \in \text{Crit}.$$

If the sum of Yang-Mills-Higgs energies of (A_∞, u_∞) and (A'_∞, u'_∞) agrees with $\langle [\omega - \mu], B \rangle$, the limit of the sequence is in \mathcal{N}_Γ for $\Gamma = \Gamma_0 \sqcup \Gamma_1$.

Step 6. (Energy loss at the $+\infty$ end in the limit) If the sum of Yang-Mills-Higgs energies of (A_∞, u_∞) and (A'_∞, u'_∞) is less than $\langle [\omega - \mu], B \rangle$, then

$$\nu = \langle [\omega - \mu], B \rangle - E(A_\infty, u_\infty) - E(A'_\infty, u'_\infty) \geq \hbar.$$

In this case, we loss some energy at the $+\infty$ end in the limit, we repeat Steps 3-4 to get a limit in \mathcal{N}_Γ with a chain of trees of length $m \geq 2$. This same process will stop after a finitely many steps due to the fact that each tree in the chain carries at least \hbar energy. This completes the compactification of $|\mathcal{N}_\Sigma(X, P, B)|$.

The compactification of $|\mathcal{N}_{\Sigma_P}(X, P, B; \{(g_i)\}_{i=1, \dots, k})|$ can be obtained in the similar manner. \square

6. OUTLOOK

In this paper, we mainly discuss the L^2 -moduli space of symplectic vortices on a Riemann surface with cylindrical end. The analysis can be generalised to the case of a family of Riemann surfaces with cylindrical end. Then we get a moduli space of L^2 -symplectic vortices fibered over the moduli space of complex structures. In particular, for a Riemann surface

$$\Sigma_{g,k} = (\Sigma, (z_1, \dots, z_k), j)$$

of genus g and with k -marked points, when $2 - 2g - k < 0$, we can consider $\Sigma_{g,k}$ as a Riemann surface of genus g and with n -punctures. By the uniformization theorem, for each complex structure on $\Sigma_{g,k}$, there is a unique complete hyperbolic metric on the corresponding punctured surface. This defines a canonical horodisc structure at each puncture, see [6]. This horodisc structure at each puncture is also called a hyperbolic cusp. Using the canonical horodisc structure at each point defined the complete hyperbolic metric on the punctured $\Sigma_{g,k}$, we can identify

the moduli space $\mathcal{M}_{g,k}$ with the moduli space of hyperbolic metrics with a canonical horodisc structure at each punctured disc. Each horodisc can be equipped with a canonical cylindrical metric on the punctured disc. In particular, we get a smooth universal family of Riemann surface with k cylindrical ends over the moduli space $\mathcal{M}_{g,k}$. Then the analysis in this paper on the L^2 -moduli space of symplectic vortices can be carried over to get a continuous family of Fredholm system defined by the symplectic vortex equations. The corresponding L^2 -moduli spaces of symplectic vortices without and with prescribed asymptotic data will denoted by

$$\mathcal{N}_{g,k}(X, P, B) \quad \text{and} \quad \mathcal{N}_{g,k}(X, P, B; \{(g_i)\}_{i=1, \dots, k})$$

respectively. Then we have the similar compactness result for this L^2 -moduli space where the index set $\mathcal{S}_{\Sigma; B}$ is replaced by $\mathcal{S}_{g,k; B}$ where the root of a principal part of each web is replaced a dual graph as in the Gromov-Witten moduli space with weights in $H_2^G(X, \mathbb{Z})$ at each vertex, each vertex carries bubbling trees (with weights in $H_2(X, \mathbb{Z})$) and each tail is assigned a chain of trees.

In the subsequence paper, we shall also establish a weak orbifold Fredholm system and a gluing principle for the compactified moduli space $\overline{\mathcal{N}}_{g,k}(X, P, B; \{(g_i)\}_{i=1, \dots, k})$ so that the virtual neighbourhood technique developed in [7] can be applied to define a Gromov-Witten type invariant from these moduli spaces([8]). We will show that the compactified moduli space $\overline{\mathcal{N}}_{g,k}(X, P, B; \{(g_i)\}_{i=1, \dots, k})$ admits an oriented orbifold virtual system and the virtual integration

$$\int_{\overline{\mathcal{N}}_{g,k}(X, P, B; \{(g_i)\}_{i=1, \dots, k})}^{vir} : H^*(I\mathcal{X}_0, \mathbb{R})^k \rightarrow \mathbb{R}$$

is well-defined. Here $I\mathcal{X}_0$ is the inertial orbifold of the symplectic reduction $\mathcal{X}_0 = \mu^{-1}(0)/G$. The Gromov-Witten type invariant is defined to be

$$\langle \alpha_1, \dots, \alpha_k \rangle_{g,k,B}^{\ell HGW} = \int_{\overline{\mathcal{N}}_{g,k}(X, P, B; \{(g_i)\}_{i=1, \dots, k})}^{vir} \partial_{\infty}^* (\pi_1^* \alpha_1 \wedge \dots \wedge \pi_k^* \alpha_k)$$

for any k -tuple of cohomology classes

$$(\alpha_1, \dots, \alpha_k) \in H^*(\mathcal{X}_0^{(g_1)}, \mathbb{R}) \times \dots \times H^*(\mathcal{X}_0^{(g_k)}, \mathbb{R}).$$

Here $\pi_i : \text{Crit}^k \rightarrow \text{Crit} = I\mathcal{X}_0$ denotes the projection to the i -th component. We emphasize that this is an invariants on $H_{CR}^*(\mathcal{X}_0)$ rather than on $H_G^*(X)$. It is different from usual HGW invariants. We call the invariant L^2 -Hamiltonian GW invariants (abbreviated as ℓHGW). In particular, when $(g, k) = (0, \geq 3)$, the above invariant can be assembled to get a new (big) quantum product $*^{HR}$ on $H^*(I\mathcal{X}_0, \mathbb{R})$. Here HR stands for Hamiltonian reduction. In a separate paper([9]), we will introduce an augmented symplectic vortex equation to define an equivariant version of this invariant on $H_G^*(X)$ when G is abelian. This enables us to define a quantum product $*_G$ on $H_G^*(X)$. We investigate its relation to $*^{HR}$, in particular, we combine symplectic vortex equation with the augmented one to define the quantum Kirwan map Q_{κ} and show that Q_{κ} is a ring morphism with respect to $*_G$ and $*^{HR}$.

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